» Introduction to Methods of Solving Quadratics

Topic 11 – Solution of Quadratic Equations

Program Items: 11(i), (iii)

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1. Factorising Monic Quadratics

You have had ample practice at solving monic quadratics in general form ($ax^2 + bx + c$, where $a = 1$). These can be solved by identifying a pair of numbers that add together to give **b**, and that multiply together to give **c**. For instance, given the following quadratic equation:

$$x^2 + 5x + 6 = 0$$

Two numbers that add to give **5** and add together to give **6** are 2 & 3. Using these allows us to factorise the expression:

$$(x + 2)(x + 3) = 0$$
$$x = -2, -3$$

You should train yourself in recognising the solutions to monic quadratics, until these become second nature. If factorising monic quadratics presents major difficulties to you, the rest of this topic will be a massive challenge.

2. Factorising Non-Monic Quadratics

This year, the complexity of quadratics you will face includes non-monic quadratics, where the co-efficient of x^2 is not 1. Again thinking in general form, unless b & c can be divided by a (e.g. $2x^2 + 10x + 12$), these quadratics require a slightly different process to be solved. Given a non-monic quadratic in general form ($ax^2 + bx + c$), we must identify a pair of numbers that add together to give **b** (this part is the same), and that multiply together to give **ac** (this part is different). So, for instance, given the following quadratic:

$$5x^2 - 12x + 4 = 0$$

Two numbers that add to give **20** (5 × 4) and add together to give **–12** are –2 & –10. We'll call these two numbers s_1 and s_2. Once we have these two numbers, we can choose one of two ways to produce the factorisation and then the solution.

» CASE 1

Factorise the quadratic in the following way:

$$(ax + s_1)(ax + s_2) \div a = 0$$

This is similar to the method for solving monic quadratics. The reason why we must divide by **a** is because expanding $(ax + s_1)(ax + s_2)$ would not give us our original quadratic ($ax^2 + bx + c$); in fact, it would give us $a^2x^2 + abx + acx$. When factorising in this way, if we chose s_1 and s_2 correctly, we should find that one of our factors will cancel out the **a.** You will see what I mean when we go through the example:

$$5x^2 - 12x + 4 = 0$$
$$(5x - 2)(5x - 10) \div 5 = 0$$
$$(5x - 2)(x - 2) = 0$$
$$x = \frac{2}{5}, 2$$

This is not the only way we can use s_1 and s_2 to solve the quadratic.

» CASE 2

Remembering that we chose s_1 and s_2 because $s_1 + s_2 = b$, we can re-arrange our quadratic in the following way:

$$ax^2 + bx + c = 0$$
$$ax^2 + s_1x + s_2x + c = 0$$

Again, if we chose s_1 and s_2 correctly, we should find that we can simply pair up the terms and factorise. This can be seen as we go through the example:

$$
\begin{aligned}
5x^2 - 12x + 4 &= 0 \\
5x^2 - 10x - 2x + 4 &= 0 \\
5x(x - 2) - 2(x - 2) &= 0 \\
(5x - 2)(x - 2) &= 0 \\
x &= \tfrac{2}{5}, 2
\end{aligned}
$$

It is also worth noticing that the order of s_1 and s_2 makes no difference to our solution:

$$
\begin{aligned}
5x^2 - 12x + 4 &= 0 \\
5x^2 - 2x - 10x + 4 &= 0 \\
x(5x - 2) - 2(5x - 2) &= 0 \\
(x - 2)(5x - 2) &= 0 \\
x &= \tfrac{2}{5}, 2
\end{aligned}
$$

These are the two ways to solve a non-monic quadratic by factorising. However, while it is easily the fastest method of solving quadratics, factorising is limited in its usefulness because it only works if the solutions are rational. If the solution is irrational (i.e. it involves surds), factorising will be of no use – you will find yourself staring at the quadratic unable to come up with any pair of numbers that add together to give b, and that multiply together to give ac. If this is the case, we must turn to other methods.

3. Completing the Square

Though disliked by some, this method is still useful in that it enables us to solve quadratics that cannot be factorised (which is a lot of them!). It takes advantage of the fact that when a quadratic ($ax^2 + bx + c$) is a perfect square, the numbers b & c are related to each either in the following way:

$$
\begin{aligned}
(x + p)^2 &= 0 \\
x^2 + 2px + p^2 &= 0
\end{aligned}
$$

If we let $q = 2p$, we see the following:

$$
\begin{aligned}
x^2 + 2px + p^2 &= 0 \\
x^2 + qx + \left(\frac{q}{2}\right)^2 &= 0
\end{aligned}
$$

Thus we can see that if we have a quadratic in the form $x^2 + qx$, we can "complete the square" by adding **the square of half of q.** Then we can convert the quadratic into the form $(x + p)^2$, which will allow us to solve it. This is all much easier to understand by using an example – we will use the earlier one so that we can anticipate the solution and know if we are on the right track or not. Beginning with this:

$$5x^2 - 12x + 4 = 0$$

We want the left-hand side to be in the form $x^2 + bx$. Thus we will move **c** to the right-hand side and then divide both sides through by **a**.

$$
\begin{aligned}
5x^2 - 12x &= -4 \\
x^2 - \frac{12}{5}x &= -\frac{4}{5}
\end{aligned}
$$

At the moment, things look worse than when we started, but now we will complete the square and things will begin to simplify. To do this we must **add the square of half of $\frac{-12}{5}$.** Half of $\frac{-12}{5}$ is $\frac{-6}{5}$; the square of this is $\frac{36}{25}$. Adding this to both sides is the crucial step that will allow us to factorise.

$$x^2 - \frac{12}{5}x = -\frac{4}{5}$$
$$x^2 - \frac{12}{5}x + \frac{36}{25} = -\frac{4}{5} + \frac{36}{25}$$
$$(x - \frac{6}{5})^2 = -\frac{20}{25} + \frac{36}{25}$$
$$(x - \frac{6}{5})^2 = \frac{16}{25}$$
$$x - \frac{6}{5} = \pm\sqrt{\frac{16}{25}} \quad *$$
$$x - \frac{6}{5} = \pm\frac{4}{5}$$
$$x = \frac{6}{5} \pm \frac{4}{5}$$
$$x = \frac{2}{5}, 2$$

Success! Notice that we have ended up with rational solutions anyway, because we used our earlier example. If the solutions to a quadratic are rational and we use this method, then at the line indicated by an asterisk (*), we should find a perfect square underneath the square root sign. However, when this method is used on a question with irrational solutions, there will be no perfect square and the irrational term will simply remain in our solution.

4. The Quadratic Formula

You will notice that completing the square took much longer than factorising. This is to be expected, since finding a rational (non-surd) solution should be much simpler than finding an irrational (surd) solution. The complexity can't be avoided – but mathematicians have sought to make the solving process easier by encapsulating the technique into a formula – the aptly-named quadratic formula. The clear advantage is that there are no 'steps' to remember when solving with this method – you simply write out the formula, substitute in the co-efficients (again, based on a quadratic in general form, $ax^2 + bx + c$), and then start simplifying until you arrive at the end. The difficulty is that you must memorise an awkward formula. However, if you can get past this initial hurdle, you will find the formula a simple and effective means of solving **any** quadratic placed in front of you. If this formula can't find a solution to your quadratic, then a solution doesn't exist! Here is the formula in its entirety:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

And here is how it works out with the example we have been using:

$$5x^2 - 12x + 4 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(5 \times 4)}}{2 \times 5}$$
$$x = \frac{12 \pm \sqrt{144 - 4(20)}}{10}$$
$$x = \frac{12 \pm \sqrt{144 - 80}}{10}$$
$$x = \frac{12 \pm \sqrt{64}}{10}$$
$$x = \frac{12 \pm 8}{10}$$
$$x = \frac{4}{10}, \frac{20}{10}$$
$$x = \frac{2}{5}, 2$$

For many, due to its simplicity, it is easiest to dive into the quadratic formula whenever an quadratic equation is presented. While you won't get the wrong answer (if you do your algebra correctly!), the drawback to this is that you will potentially waste a lot of time going through the motions of the quadratic formula, when you could have reached the solution much faster by factorising (as in the example we just used). As stated earlier, you must train yourself to recognise when a quadratic can and can't be factorised – this is the hardest part of identifying which solving method you should use.