» Synthetic Division

With persistence and practice, long division becomes the 'old faithful' method of dividing through polynomials. The problem is that while it's old and faithful, it's also **slow** and faithful. *Long division* is so named because it takes a *long time.* Not only that, it is also prone to errors, and an error that appears early in the long division hams up the entire answer. So mathematicians devised a method that came to be called *synthetic division*.

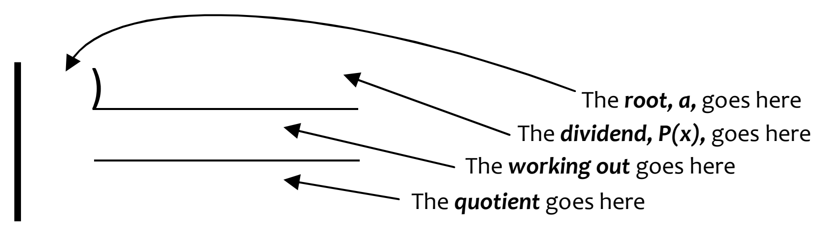
Synthetic division carries out exactly the same mathematical processes as long division, which is how (as we will see) it arrives at the same answer. However there are several key differences:

a. It eliminates the 'record-keeping' of long division – whereas all the multiplying and subtracting steps are documented in long division, synthetic division removes all this (making it more concise, but more difficult to track down errors if you make them).
b. Since people are (usually) instinctively better at addition than subtraction, synthetic division replaces the subtraction operation with addition to reduce the chance of errors appearing.
c. It only works when the divisor is a **linear monic binomial** – that is, a polynomial with two terms (binomial), whose leading co-efficient is 1 (monic) and whose degree is 1 (linear). In other words, we want something in the form $(x - a)$. This might seem like a drastic shortcoming – after all, long division can divide through by any polynomial under the sun. But it just so turns out that dividing through by factors like $(x - a)$ is exactly what we want to do in this case, so this is no problem for us.
d. By convention synthetic division doesn't signify the indices – or even the variables – of the divisor or dividend. This is the main reason why (as you will see) synthetic division looks so strange to begin with – at first appearance, the 'working out' for synthetic division just seems like a random arrangement of numbers. But once you become accustomed to the method, you'll be fine.

For all the reasons outlined above, synthetic division is *fast*. Many students are apprehensive at having to learn a new method for doing something they are already able to do (i.e. long division). That apprehension usually fades once you see how much time and effort synthetic division will save you. So let's get started!

» Step 1: DRAW up the table

One of the drawbacks of this method is that it can be very confusing to grasp initially. So I'm going to go through each of the steps real slowly to make sure you catch all the details. The first thing you need to do is draw the table – which is simple enough. By convention, it looks like an upside-down long division table.



» Step 2: Write the ROOT

Next, we want to fill out the top line of the table, which is where the details of the question go in: what do we want to divide, and what will we divide by? We'll use the example we looked at earlier, so that we can compare the speed of the methods. $P(x) = -2x^3 - x^2 + 16x + 15$. Since synthetic division replaces long division, we will still need to guess and check to find the first factor. Thus we can discover that $P(-1) = 0$. What you need to first realise is that the value of a **represents the root for which $x - a$ is a factor.** In other words, since $(x + 1)$ is a factor of $P(x)$, $x = -1$ is a root of the same polynomial. The converse is also true.

In synthetic division, instead of writing the *factor* at the front (as in long division), we simply write the root. The reason for this will become clear soon enough.

$$\begin{array}{r|l} -1 & \underline{\qquad\qquad\qquad} \\ & \underline{\qquad\qquad\qquad} \end{array}$$

» Step 3: Write the DIVIDEND

The third step is to write in the polynomial that we are dividing by. As mentioned above in the list of differences between long and synthetic division, we are only going to write the *co-efficients* of the polynomial, and leave out the variables. As with long division, it's vitally important that you write the co-efficients in the order of their index, from greatest to least – so if the polynomial is not given to you in that form (for instance, $P(x) = 15 - 2x^3 + 16x - x^2$) then you must re-arrange it so that it is. Then, simply write down the numbers!

$$\begin{array}{r|rrrr} -1 & \underline{-2} & \underline{-1} & \underline{16} & \underline{15} \\ & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{array}$$

Again, as with long division, if there is a term 'missing' (for instance, the x^2 in $F(x) = 4x^3 - 3x + 1$) then we must indicate it with a 0. If we were dividing through $F(x)$ instead of $P(x)$, it would look like this:

$$\begin{array}{r|rrrr} -1 & \underline{4} & \underline{0} & \underline{-3} & \underline{1} \\ & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{array}$$

» Step 4: Write the LEADING CO-EFFICIENT

This step is so simple it almost isn't worth putting separately – but I said I would go through this process slowly! So what we do now is write the *leading co-efficient of the dividend* (in this case, -2) at the bottom, on the third line.

$$\begin{array}{r|rrrr} -1 & -2 & -1 & 16 & 15 \\ & & & & \\ \hline & -2 & & & \end{array}$$

» Step 5: MULTIPLY by the root

Now the real work begins. Take the co-efficient you just wrote down (-2) and *multiply it by the root* (in this case, -1). Write that result on the second row, underneath the next co-efficient.

$$\begin{array}{r|rrrr} -1 & -2 & -1 & 16 & 15 \\ & & 2 & & \\ \hline & -2 & & & \end{array}$$

» Step 6: ADD

As we mentioned earlier, subtraction is replaced by addition in synthetic division. This is possible because we are doing our operations with the *root* (-1) instead of the *factor* ($x + 1$). So the next step is to *add the numbers in the next column and write the next co-efficient.* $-1 + 2 = 1$, so we write 1 on the third line.

$$\begin{array}{r|rrrr} -1 & -2 & -1 & 16 & 15 \\ & & 2 & & \\ \hline & -2 & 1 & & \end{array}$$

» Step 7: Rinse and REPEAT

Steps 5 and 6 must now be repeated until we run out of numbers. Multiply the co-efficients by the root each time and then add these together to get the next co-efficient. Here's what you should end up with:

$$\begin{array}{r|rrrr} -1 & -2 & -1 & 16 & 15 \\ & & 2 & -1 & -15 \\ \hline & -2 & 1 & 15 & 0 \end{array}$$

» Step 8: TRANSLATE the solution

The last step is to *translate* that mess of numbers into the actual solution, co-efficients and all. Notice that since we are dividing through by a *linear polynomial* (degree 1), the degree of the quotient should be one less than the degree of dividend. In other words, since $P(x) = -\mathbf{2x^3} + something$, then the quotient ought to be $Q(x) = -\mathbf{2x^2} + somethingelse$. Knowing this, write down the terms of the quotient with all the variables added in, on the fourth line:

$$\begin{array}{r|rrrr} -1 & -2 & -1 & 16 & 15 \\ & & 2 & -1 & -15 \\ \hline & -2 & 1 & 15 & 0 \\ & -2x^2 & +\, x & +\, 15 & \end{array}$$

As you've probably guessed, the last number in the second line is the remainder. Thankfully, in this case it is 0 – meaning our working is correct!

You may right now be thinking: "*Seriously, 8 steps?! This is insane.*" This is what I used to do and in high school, I actually gave up on synthetic division because it was never explained to me clearly.

In answer to that attitude, I say two things. Firstly, some of the steps are really trivial and take no time at all (e.g. step 1, step 4). Secondly, the remaining steps – once you become used to doing them and the order that they are done in – are much faster and more accurate than the corresponding steps in long division. To prove this, all I ask of you is to learn this method and give it a try. Use the exercises on the next page to practise.

» Exercises

Divide through the following polynomials by the given factor, and then use the result to produce all the solutions (roots) and factorisations.

1. $8x^3 + 18x^2 - 47x - 42 = 0; (x - 2)$
2. $3x^3 - 4x^2 - 49x - 30 = 0; (x + 3)$
3. $-15x^3 - 13x^2 + 30x - 8 = 0; (x + 2)$
4. $-14x^3 + 69x^2 - 90x + 27 = 0; (x - 3)$

» Answers

First I give the factorisation, and then the accompanying solutions (roots).

1. $(x - 2)(2x + 7)(4x + 3); x = 2,\ -3\frac{1}{2},\ \frac{-3}{4}$
2. $(x + 3)(x - 5)(3x + 2); x = -3,\ \frac{-2}{3},\ 5$
3. $(x + 2)(3x - 1)(4 - 5x); x = -2,\ \frac{1}{3},\ \frac{4}{5}$
4. $(x - 3)(2x - 3)(3 - 7x); x = \frac{3}{7},\ 1\frac{1}{2},\ 3$