» Solving Polynomials

We have already looked at *Solving Quadratics*, and noticed that there are several methods that we can apply to solve equations of the form $ax^2 + bx + c = 0$. Factorising, completing the square and the quadratic formula were the three main strategies we could use to find values of x to satisfy quadratics.

Quadratics are a specialised type of polynomial. Namely, they are polynomials with order (or degree) 2. In this topic we have been investigating methods of solving polynomials of degrees greater than 2. But solving polynomials of higher order is considerably more difficult, and requires us to lay some foundations first.

» Why we don't learn the Cubic Formula

"Solving polynomials of higher order is considerably more difficult." It is one thing to accept this statement on face value, and another thing altogether to see this additional difficulty illustrated. Therefore let me try to give you a glimpse of how increasing the order of a polynomial increases how difficult it is to solve.

Consider a polynomial of degree 1 – that is, a **linear** equation – in the form $ax + b = 0$. The formula to solve this with respect to x would simply be $x = \frac{-b}{a}$. A simple polynomial is simple to solve.

Now consider a polynomial of degree 2 – that is, a **quadratic** equation – in the form $ax^2 + bx + c = 0$. By now we know that the formula to solve this is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. You can see that it has become drastically more complicated – though it is still manageable, and most of us (with some time and effort) have no trouble memorising the quadratic formula.

Now, consider a polynomial of degree 3 – that is, a **cubic** equation – in the form $x^3 + ax^2 + bx + c = 0$. The solutions for x are given by this gargantuan formula:

$$x = \frac{\frac{1}{3}\left(\frac{a^2}{3} - b\right)}{\sqrt[3]{-\frac{1}{2}\left(c + \frac{2a^3 - 9ab}{27}\right) \pm \sqrt{\frac{1}{4}\left(c + \frac{2a^3 - 9ab}{27}\right)^2 + \frac{1}{27}\left(b - \frac{a^2}{3}\right)^3}}} - \frac{a}{3} + \sqrt[3]{-\frac{1}{2}\left(c + \frac{2a^3 - 9ab}{27}\right) \pm \sqrt{\frac{1}{4}\left(c + \frac{2a^3 - 9ab}{27}\right)^2 + \frac{1}{27}\left(b - \frac{a^2}{3}\right)^3}}$$

No joke. If you want to solve a cubic equation through the use of a formula, this is the best formula mathematicians have come up with. This particular method was published by a guy named Cardano in 1545, and no one has been able to improve upon it since. Not only that, if you pay attention to the form of the cubic equation above, you'll notice that this formula only works for monic cubic equations – the leading co-efficient must be 1. To add insult to injury, this formula doesn't even work 100% of the time – cubic equations with a particular set of mathematical qualities will cause the formula to fail.

I hope that this is enough to explain why I am not even going to bother quoting the formula for solving quartic equations (polynomials of degree 4). I'm also not going to quote the formula for solving quintic equations (polynomials of degree 5), because one doesn't currently exist – and mathematicians have proved that there never will (look up *Abel's impossibility theorem* if you want to see why).

The point here has been to illustrate that when it comes to solving polynomials, practically speaking we must abandon the use of formulae. They're useful for quadratics, and that's basically it. So we need to search for another, easier way of solving polynomials. Unfortunately, we don't have the time to sit around like the ancient Greeks just wearing togas and pondering mathematical possibilities. So instead of going on a search, we're going to go on a guided tour of the mathematics relevant to solving polynomials.

» Foundational Theorems

Our tour begins at a strange place – long division. It's strange because long division is about dividing constant integers – polynomials and their variable terms are nowhere in sight. But they will appear before long.

» Painting with numbers

Let's remind ourselves of how long division works by looking at an example. Consider the question: what is 3041 divided by 17? Here's how it works out:

$$\begin{array}{r} 178\ R15 \\ 17\overline{)3041} \\ \underline{17} \\ 134 \\ \underline{119} \\ 151 \\ \underline{136} \\ 15 \end{array}$$

In other words, 17 goes 178 times into 3041, leaving a remainder of 15.

What's that notation?

Before you get any further, it's worth clarifying some of the notation I'm using in this document. When I write something like ***P(x)***, I'm referring to a function called ***P***, that has variables called ***x***. So ***u(y)*** would be a function called ***u*** that has variables called ***y***.

This notation is used all the time in maths and is very general. For example, I could even talk about ***Q(a,b,c)*** – a function called ***Q*** that has three different variables called ***a***, ***b*** & ***c*** – you guessed it, the quadratic formula.

» Just add variables and stir

Now, can we use this tool to divide polynomials through by other polynomials? What if we want to divide a polynomial, $P(x) = x^2 + 5x + 10$, by the divisor $d(x) = x + 2$?

$$\begin{array}{r} ? \\ x+2\overline{)x^2+5x+10} \end{array}$$

Trying this process out with polynomials is strange because all the steps in long division rely on concrete numbers, and knowing how many times the divisor goes into the various parts of the dividend. In polynomial division, we are dividing terms into each other that vary in size. This is the reason why, when we divide through, we simply divide through by the *leading term* of the divisor and 'ignore' the size of the other terms – because the leading term is the most significant component in the division. So this is how it works out:

$$
\begin{array}{r}
x+3 \; R4 \\
x+2\overline{)x^2+5x+10} \\
\underline{x^2+2x} \\
3x+10 \\
\underline{3x+6} \\
4
\end{array}
$$

Thus the result is that $(x^2 + 5x + 10) \div (x + 2) = (x + 3)\ R4$. Now, we can express the result of this long division in the following way: $P(x) = (x + 2)(x + 3) + 4$. Notice that $P(-2) = 4$, the remainder.

Let's generalise this equation. Suppose:

$$
\begin{array}{r}
f(x) \;\; R \\
x-a\overline{)P(x)}
\end{array}
$$

$$\therefore P(x) = (x - a)f(x) + R$$

$$P(a) = (a - a)f(a) + R$$

$$\boldsymbol{P(a) = R}$$

This result is known as the *Remainder Theorem*, since it describes the significance of the remainder when dividing through by $x - a$.

» A more useful application

This is great if we want to find remainders quickly. But what if we want to find an actual *factor* that divides properly into $P(x)$ without leaving a remainder? We set $P(a) = 0$. If we can *guess and check* a value of a that satisfies this equation, then $(x - a)$ is a factor of $P(x)$. This special variant of the Remainder Theorem is known as the *Factor Theorem,* since it is used in locating factors to polynomials.

Once a factor has been successfully identified, we can divide through the original polynomial by this newly discovered factor. The resultant polynomial will be of lesser order, and thus easier to solve.

For instance, suppose $P(x)$ is a cubic polynomial (order 3) and we identify one of its factors to be $(x - 3)$ (order 1). The quotient of $P(x) \div (x - 3)$, $Q(x)$, will be a quadratic (when you divide order 3 by order 1, you get order 2 – remember your index laws!). We know how to solve quadratics – and so we have effectively reduced the difficulty of the problem we are attempting to solve.