» Generating Solutions for Cubics

This document will guide you, step-by-step, through the process for solving cubic polynomials. Consider the following example question:

Find all the roots of the polynomial $P(x) = -2x^3 - x^2 + 16x + 15$.

» Step 1: LOCATE the first factor

The first step is, sadly, unavoidably laborious. We need to *guess and check* until we find a factor of $P(x)$. This process would be prohibitively slow if all we could do was long division (e.g. dividing $P(x)$ by some divisor $d(x)$ and checking to see if the remainder is 0). However, with the aid of the Factor Theorem, all we need to do is substitute numbers into $P(x)$ until $P(a) = 0$, which is a (relatively) simple process. Then we will have located a factor, namely $(x - a)$. It is worth noting that you will never be asked to solve a polynomial in this manner where a factor will not emerge quickly if you try values of a that are close to 0 (both positive and negative). Here's how it plays out:

> Try $a = 1$.
> $P(1) = -2(1)^3 - (1)^2 + 16(1) + 15$
> $\quad = -2 - 1 + 16 + 15$

Let me point out two things here. Firstly, I didn't start with $a = 0$ because I would have simply ended up with the *constant term* of the polynomial, which in this case is 15 – so that wouldn't produce a factor. Secondly, I am actually going to stop my calculations above and move onto a new value of a because it's already obvious that $a = 1$ won't produce a factor either. So we continue to search:

> Try $a = -1$.
> $P(-1) = -2(-1)^3 - (-1)^2 + 16(-1) + 15$
> $\quad = 2 - 1 - 16 + 15$
> $\quad = 0$
> $\therefore (x + 1)$ is a factor of $P(x)$.

» Step 2: DIVIDE through by the factor

Now we want to divide $P(x)$ through by the factor we have just found. See over the page for the working with long division. Alternatively, you can divide through by synthetic division, which is trickier but faster. Ask a friend (or me, if your friend doesn't know) to help you out with synthetic division if you've forgotten how to do it or if you just find it difficult – the time learning it will be worth it.

$$
\begin{array}{r}
-2x^2 + x + 15 \\
x+1 \overline{)\, -2x^3 - x^2 + 16x + 15} \\
\underline{-2x^3 - 2x^2 } \\
x^2 + 16x \\
\underline{x^2 + x } \\
15x + 15 \\
\underline{15x + 15} \\
0
\end{array}
$$

As we predicted (by using the Factor Theorem), the remainder is 0. The quotient is what we are after: now we know that $P(x)$ can be expressed in the form $(x+1)(-2x^2+x+15)$. This leads onto the final step.

» Step 3: SOLVE the remaining quadratic

This is really the easy part, because we have already spent weeks on getting good at this. Since we already have one of the cubic's three solutions (in this case, $x = -1$), all that remains is to solve the remaining quadratic (in this case, $-2x^2 + x + 15$) by whatever means you like – factorise, complete the square or use the quadratic formula. The result will have three possible outcomes:

i. If the original cubic had **three solutions**, then solving the quadratic will give us the last two.
ii. If the original cubic had only **two solutions** (e.g. $x^3 - x^2 - x + 1$), then solving the quadratic will give us the one we don't already have.
iii. If the original cubic had only **one solution** (e.g. $x^3 + 6x^2 + 12x + 8$), then we will find that the quadratic has no real solutions – that is to say, the original solution we located is the only one.
iv. *All cubic equations have at least one solution.* I'll leave you to work out why there are no cubics with zero solutions! Here's a hint: try graphing a cubic without solutions (i.e. roots). You'll have a lot of trouble.

Since you're so familiar with solving quadratics, I'm not going to include the steps here. You do your own working; you should end up with the two solutions of the quadratic being $x = -2\frac{1}{2}$ and $x = 3$.

Hence, the three roots of $P(x) = -2x^3 - x^2 + 16x + 15$ are $x = -2\frac{1}{2}, -1$ & 3.

You may also be asked to produce the factorisation of a polynomial – these are really equivalent questions, involving all the same steps, but phrasing the answer in a different way. In this case, the correct factorisation is $P(x) = (x+1)(2x+5)(3-x)$. QED.