» Locating Oblique Asymptotes

We have noticed that functions like $y = \frac{1}{x}$ have *asymptotes* that dictate how they behave close to positive and negative infinity. Horizontal and vertical asymptotes are relatively easy to locate. But what about oblique asymptotes? These are a little trickier.

» Look for the degree

The first thing to notice is that, assuming a function can be expressed as a fraction, oblique asymptotes occur **when the degree of a function's numerator is greater than the degree of the function's denominator**. For instance, in the function $y = \frac{x^2+1}{x}$, the numerator has a degree of 2 and the denominator has a degree of 1.

» "Remove" the asymptote

Secondly, the oblique asymptote can be identified because **it can be "removed" from the fraction and made to stand on its own** in the function, such that whatever remains tends toward zero. For instance, in the function mentioned before $\left(y = \frac{x^2+1}{x}\right)$, I can simply divide the numerator through by the denominator, which leaves me with $y = x + \frac{1}{x}$. We have "removed" the oblique asymptote (which in this case is $y = x$) and made it stand on its own in the function, apart from the fraction where it came from. Why is this useful? How does it help us see the way the function behaves as x approaches infinity (or negative infinity)?

Here is the key idea to comprehend. Since $\frac{1}{x} \to 0$ as $x \to \infty$, $y \to x$ as $x \to \infty$. Think this through and make sure that this part makes sense to you, as it is fundamental to the rest of what we're going to do. If you don't understand this step, you must immediately ask someone (like me, or a friend) to explain it to you before you go any further!

» Use polynomial division

However, you can't always simply divide the numerator through by the denominator. Take, for instance, $y = \frac{x^2+5}{x-2}$. Clearly, this fits our criteria for an oblique asymptote – the degree of the numerator is 2 while the degree of the denominator is 1. But $x - 2$ does not divide evenly into $x^2 + 5$. This, therefore, brings us to the third point, which is that the process of "removing" the oblique asymptote from the fraction can be **generalised by the process of polynomial division.** Choose your flavour – long or synthetic – and then divide the numerator by the denominator.

» An example

For instance, $(x^2 + 5) \div (x - 2) = (x + 2)$, with a remainder of 9. This means that we can rewrite $x^2 + 5$ as $(x - 2)(x + 2) + 9$. Notice then what we are able to do when we substitute this into the numerator of the function above.

$$
\begin{aligned}
y &= \frac{x^2+5}{x-2} \\
&= \frac{(x-2)(x+2)+9}{x-2} \\
&= \frac{(x-2)(x+2)}{x-2} + \frac{9}{x-2} \\
&= (x+2) + \frac{9}{x-2}
\end{aligned}
$$

By using polynomial division, we have rearranged the function to a point where we were able to remove the oblique asymptote out. We know that $x + 2$ must be the oblique asymptote, because $\frac{9}{x-2} \to 0$ as $x \to \infty$.

Use this process to locate the oblique asymptotes for the following functions, and hence sketch them neatly on separate sets of axes. N.B. There are sometimes alternative methods for identifying exactly what the oblique asymptote is – have a think about what you might do instead of polynomial division!

» Questions

1. $y = \frac{x^2-2x-15}{x+2}$
2. $y = \frac{2x^2+x+2}{x+1}$
3. $y = \frac{2x^2+3x-7}{4x-2}$
4. $y = \frac{x^3+x^2-x+2}{x+1}$
5. $y = \frac{-x^3+x^2+4x-6}{x-1}$

» Answers

Q	Function	Factorised Numerator	Oblique Asymptote
1	$y = \frac{x^2-2x-15}{x+2}$	$y = \frac{(x-4)(x+2)-7}{x+2}$	$y = x - 4$
2	$y = \frac{2x^2+x+2}{x+1}$	$y = \frac{(x+1)(2x-1)+3}{x+1}$	$y = 2x - 1$
3	$y = \frac{2x^2+3x-7}{4x-2}$	$y = \frac{\left(\frac{1}{2}x+1\right)(4x-2)-5}{4x-2}$	$y = \frac{1}{2}x + 1$
4	$y = \frac{x^3+x^2-x+2}{x+1}$	$y = \frac{(x^2-1)(x+1)+3}{x+1}$	$y = x^2 - 1$
5	$y = \frac{-x^3+x^2+4x-6}{x-1}$	$y = \frac{(4-x^2)(x-1)-2}{x-1}$	$y = 4 - x^2$