

» Deriving the Index Laws

- de·rive** (*verb*)
1. to trace from a source or origin.
 2. to reach or obtain by reasoning; deduce; infer.

Index notation has risen out of the desire for a faster and more intuitive way of writing numbers that clarifies the connection between multiplication, division and surds. With this new form of notation comes a whole series of new 'laws' that govern the way numbers interact when they are written in such a way. It doesn't take too much effort to learn the *index laws* by rote, but you will understand them better by having a brief look at where the laws came from to begin with.

» Indices are Abbreviations

Though it might not seem like it to begin with, it's a well-publicised fact that mathematicians are lazy – they are always searching for faster ways to do things. This is why we use *multiplication notation* – instead of writing everything in sums, we write things as products instead.

$$5 + 5 + 5 + 5 = 20$$

But it is quicker to write:

$$5 \times 4 = 20$$

Multiplication is a 'shortcut' way of representing *repeated additions of the same number*. The multiplier (in this case, 4) represents the number of times that you want to add the number (in this case, 5) together.

Indices work in exactly the same way:

$$3 \times 3 \times 3 \times 3 = 81$$

But it is quicker to write:

$$3^4 = 81$$

When we give a number (called the *base*) an index (also called the *exponent* or *power*), it is called *raising to a power*, or more simply *exponentiation*. Exponentiation is a 'shortcut' way of representing *repeated multiplication by the same number*. The index (in this case, 4) represents the number of times that you want to multiply the number (in this case, 3) together. I hope you can see the clear parallel.

» The basic rule: multiplying is abbreviated to adding

Multiplication (itself an abbreviation) is being shortened into indices. Out of this basic definition flows the first rule. In the example above, the index (4) signified that we were multiplying the number (3) by itself four times. Extending this shows that when we multiply terms together that have the same base, we simply add their indices. Have a look at this example:

$$\begin{array}{l}
 5 \times 5 = 5^2 \\
 5 \times 5 \times 5 = 5^3 \\
 \therefore 5^2 \times 5^3 = 5 \times 5 \times 5 \times 5 \times 5 \\
 = 5^5
 \end{array}$$

It only takes a little observation to generalise this fact – and that is all that mathematical laws are (generalisations of mathematical fact). From the example above we can see that $a^m \times a^n = a^{m+n}$.

» Fractional indices

This is where we derive the meaning of indices that are not whole numbers. What would be the index of a number that multiplied by itself to give 5?

$$\begin{array}{l}
 5 = 5^x \times 5^x \\
 5^1 = 5^{2x} \\
 1 = 2x \\
 x = \frac{1}{2} \\
 \therefore 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5
 \end{array}$$

Notice what this implies. We asked a question about *a number that multiplied by itself to give 5*. But we already have a definition for what that number is – it's the square root of 5.

$$\begin{array}{l}
 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5 \\
 \text{But } \sqrt{5} \times \sqrt{5} = 5 \\
 \therefore 5^{\frac{1}{2}} = \sqrt{5}
 \end{array}$$

In order to help us generalise this observation into a rule (as we did before), let's look at another example:

$$\begin{array}{l}
 a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a \\
 \text{But } \sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a \\
 \therefore a^{\frac{1}{3}} = \sqrt[3]{a}
 \end{array}$$

So when an index is a fraction $\frac{1}{n}$, it represents *taking the n-th root of the base*. That is to say, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

» Zero

One might then (fairly) ask, “what does it mean to raise a number to the power of 0, then?” Our original definition doesn't seem particularly helpful, because it would indicate “multiplying by the base number 0 times”. What does that mean?

We can understand what this means by considering our basic rule (when we multiply terms with the same base, we add their indices). What happens when the index of one of those numbers happens to be 0?

$$\begin{array}{l} 5^3 \times 5^0 = 5^{3+0} \\ = 5^3 \end{array}$$

So, multiplying by a number with index 0 has no effect on the final answer – the result is simply what we started with. In our number system, there is only one number that fits this definition – that is, 1.

$$\begin{array}{l} 5^3 \times 5^0 = 5^3 \times 1 \\ = 5^3 \\ \therefore 5^0 = 1 \end{array}$$

We could multiply examples to show that this principle holds for any number we raise to the power of 0. Therefore we have another index rule on our hands: $a^0 = 1$.

» Negative indices

This helps us to understand what negative indices mean. If we multiply two numbers together whose bases are the same but whose indices are negatives of each other, the final answer should have an index of 0.

$$\begin{array}{l} 5^3 \times 5^{-3} = 5^{3+(-3)} \\ = 5^0 \\ = 1 \end{array}$$

Therefore, a negative index has the opposite effective of a positive index. A *negative index signifies division rather than multiplication*. Visually, this has the effect of ‘flipping’ a number onto the denominator if it has a negative index.

$$\begin{array}{l} 5^3 \times 5^{-3} = 5 \times 5 \times 5 \div 5 \div 5 \div 5 \\ = 1 \\ \text{Alternatively, we could write:} \\ 5^3 \times 5^{-3} = 5^3 \times \frac{1}{5^3} \\ = \frac{5^3}{5^3} \\ = 1 \end{array}$$

To generalise, we can write: $a^{-n} = \frac{1}{a^n}$. From here we can also see related truths (called *corollaries* in mathematics) like $\frac{a^m}{a^n} = a^{m-n}$.

» Stacking up indices

We can also easily demonstrate what happens when you 'stack up' indices – take a term (made up of a base and its index) and apply another index to the whole thing, like so:

$$\begin{aligned} (5^2)^3 &= 5^2 \times 5^2 \times 5^2 \\ &= 5^{2 \times 3} \\ &= 5^6 \end{aligned}$$

Very simply, $(a^m)^n = a^{mn}$.

» Raising a product to an index

What is the result when we two numbers multiplied together are both raised to an index?

$$\begin{aligned} (2 \times 7)^4 &= (2 \times 7) \times (2 \times 7) \times (2 \times 7) \times (2 \times 7) \\ &= 2 \times 2 \times 2 \times 2 \times 7 \times 7 \times 7 \times 7 \\ &= 2^4 \times 7^4 \end{aligned}$$

Equally simply, $(ab)^n = a^n b^n$.

» Exercises

1. Find x if:

- $5^x = 125$
- $16^x = 128$
- $8^{-x} = \frac{1}{32}$
- $3^{2x+1} = \frac{1}{27}$
- $5^{x-1} = \frac{1}{125}$
- $7^{x+\frac{1}{2}} = 343$
- $9^{4x-2} = 243$
- $3^{2x+2} = \frac{1}{81}$
- $3^{2x+1} = 27\sqrt{3}$
- $2^{x-1} = \frac{\sqrt{2}}{32}$

2. Find y if:

- $2^y = \frac{4^{\frac{1}{2}} \cdot 8^{-2}}{2^4}$
- $3^y = \frac{81^2 \cdot 27^{\frac{2}{3}}}{9^3}$
- $5^y = \frac{25^3 \cdot 5^{-2}}{125^{\frac{2}{3}}}$
- $3^y = \frac{243 \cdot 3\sqrt{3}}{27^{\frac{3}{2}}}$
- $2^y = \frac{32^{\frac{1}{2}} \cdot (4\sqrt{2})^{-3}}{(0.5)^3}$
- $7^y = \frac{49^{\frac{3}{2}}}{(7\sqrt{7})^2}$
- $2^y = \frac{4^{-3} \cdot 8^{\frac{2}{3}}}{16^{\frac{1}{2}}}$
- $3^y = \frac{3^{-4} \cdot 81^{\frac{1}{2}}}{27^2}$

$$\begin{aligned} \text{i. } 2^y &= \frac{4^2 \cdot 16^{\frac{3}{2}}}{32^{\frac{1}{2}}} \\ \text{j. } 5^y &= \frac{125^{-2} \cdot 25}{(0.2)^3} \\ \text{k. } 3^y &= \frac{9^{\frac{1}{2}} \cdot \sqrt{27}}{27^{\frac{4}{3}}} \\ \text{l. } 2^y &= \frac{4\sqrt{8} \cdot 16^{-\frac{1}{2}}}{64^{\frac{3}{4}}} \end{aligned}$$

3. Find a & b if:

$$\begin{aligned} \text{a. } (2 + \sqrt{3})^2 &= a + b\sqrt{3} \\ \text{b. } (3 + 2\sqrt{2})^2 &= a + b\sqrt{2} \\ \text{c. } (2\sqrt{3} + 3\sqrt{5})^2 &= a + b\sqrt{15} \end{aligned}$$

4. Find the values of p & q:

$$\begin{aligned} \text{a. } 3^{p-q} &= 27 \\ 3^{2p+q} &= 81 \\ \text{b. } 3^{p+q} &= \frac{1}{3} \\ 3^{4p+q} &= \frac{1}{27} \\ \text{c. } 4^{3p-2q} &= 16 \\ 4^{2p+3q} &= \frac{1}{4} \\ \text{d. } 2^{2p-q} &= 8 \\ 4^{p+q} &= \frac{1}{64} \\ \text{e. } 9^{2p+q} &= \frac{1}{81} \\ 3^{12p-q} &= \frac{1}{729} \\ \text{f. } 4^{3p-q} &= 2 \\ 8^{9p+q} &= 4 \\ \text{g. } 3^{2p-3q} &= \sqrt{3} \\ 5^{p-2q} &= 5\sqrt{5} \\ \text{h. } 5^{2p-q} &= 0.2 \\ 2^{p-q} &= 0.25 \end{aligned}$$

» Answers

1. x is equal to:

- a. 3
- b. $\frac{7}{4}$
- c. $\frac{5}{3}$
- d. -2
- e. -2
- f. $\frac{5}{2}$
- g. $\frac{9}{8}$
- h. -3
- i. $\frac{5}{2}$
- j. $-3\frac{1}{2}$

2.