» Simpson's Rule

We use Simpson's Rule to calculate approximate areas under curves. The basic formula is as follows:

$$A = \int_a^b f(x)dx \approx \frac{\boldsymbol{b-a}}{\boldsymbol{6}}\left[\underset{\substack{\uparrow \\ \text{lower} \\ \text{limit}}}{\boldsymbol{f(a)}} + \boldsymbol{4f}\left(\underset{\substack{\uparrow \\ \text{midpoint} \\ \text{of limits}}}{\frac{\boldsymbol{a+b}}{\boldsymbol{2}}}\right) + \underset{\substack{\uparrow \\ \text{upper} \\ \text{limit}}}{\boldsymbol{f(b)}}\right]$$

This divides up the function $f(x)$ into 2 sub-intervals. If you would like to divide up the function into more sub-intervals (for a greater degree of accuracy), a slightly different formula is used:

$$A \approx \frac{\boldsymbol{h}}{\boldsymbol{3}}[(\underset{\substack{\uparrow \\ \text{values at} \\ \text{endpoints}}}{\boldsymbol{y_1 + y_{n+1}}}) + \boldsymbol{2}(\underset{\substack{\uparrow \\ \text{odd} \\ \text{values}}}{\boldsymbol{y_3 + y_5}} + \cdots) + \boldsymbol{4}(\underset{\substack{\uparrow \\ \text{even} \\ \text{values}}}{\boldsymbol{y_2 + y_4}} + \cdots)]$$

N.B. h refers to the 'height' of each sub-interval, $\frac{b-a}{n}$ where n is the number of sub-intervals you choose.

Here is an example of how it is used in an actual question.

e.g. Use Simpson's rule to find an approximate value for $A = \int_1^2 \frac{1}{x}dx$, using:

a) 2 sub-intervals
b) 6 sub-intervals

» Solution

a) We can read the following facts straight from the question:

$a = 1$

$b = 2$

$f(x) = \frac{1}{x}$

Since we are only using 2 sub-intervals, we can use the basic formula for Simpson's rule.

$$A \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\approx \frac{2-1}{6}\left[f(1) + 4f\left(\frac{1+2}{2}\right) + f(2)\right]$$

$$\approx \frac{1}{6}\left[\frac{1}{1} + 4\left(\frac{1}{3/2}\right) + \frac{1}{2}\right]$$

$$\approx \frac{1}{6}\left[1 + \frac{8}{3} + \frac{1}{2}\right]$$

$\approx \frac{25}{36}$ units^2, which is 0.6944444 units^2 (7 decimal places)

b) Using 6 sub-intervals instead of 2 changes a few things. We need to use the more complicated formula for Simpson's rule, and we also need to calculate h.

$n = 6$ (the number of sub-intervals)

$h = \dfrac{b-a}{n} = \dfrac{1}{6}$

It will also help us to work out the function values in advance.

n	1	2	3	4	5	6	7
x_n	$\frac{6}{6} = 1$	$\frac{7}{6}$	$\frac{8}{6} = \frac{4}{3}$	$\frac{9}{6} = \frac{3}{2}$	$\frac{10}{6} = \frac{5}{3}$	$\frac{11}{6}$	$\frac{12}{6} = 2$
y_n	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$

Remember that having n sub-intervals means we have $n+1$ function values, and quote the formula:

$$A \approx \frac{h}{3}[(y_1 + y_7) + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6)]$$

$$\approx \frac{1}{18}[(y_1 + y_7) + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6)]$$

$$\approx \frac{1}{18}\left[\left(1 + \frac{1}{2}\right) + 2\left(\frac{3}{4} + \frac{3}{5}\right) + 4\left(\frac{6}{7} + \frac{2}{3} + \frac{6}{11}\right)\right]$$

$$\approx \frac{1}{18}\left[\frac{3}{2} + \frac{27}{10} + \frac{1912}{231}\right]$$

$$\approx 0.6931698 \text{ units}^2 \text{ (7 decimal places)}$$

» Note

Going through this entire process will help you appreciate the elegance and power of integration, which Simpson's rule is attempting to imitate. Compare the amount of work required to solve the original question just by integrating:

$$A = \int_1^2 \frac{1}{x}\,dx$$

$$= \Big[\ln(x)\Big]_1^2$$

$$= \ln(2) - \ln(1)$$

$$= \ln(2) \text{ units}^2, \text{which is } 0.6931472 \text{ units}^2 \text{ (7 decimal places)}$$

See how easy it was to calculate the answer! Not only that, integration is *precise* – Simpson's rule is only approximate. Notice that since we now have the exact answer, we can see the degree of accuracy that Simpson's rule achieves (and how that accuracy increases when we use a greater number of sub-intervals).