» **Exponential Growth & Decay** | Worked HSC Questions

1. A population $N(t)$ varies with time according to the equation $N(t) = Ce^{kt}$.
   a. Show that if $\{a, b \in \mathbb{R}^+ : a + b = 1\}$, then $N(at + bu) = [N(t)]^a \times [N(u)]^b$ for $\{t, u \in \mathbb{R}^+\}$.
   b. Hence, or otherwise, find $N(13)$, given that $N(3) = 10$ and $N(18) = 100$.

2. The rate of increase of a population $P(t)$ is governed by the equation $\frac{dP}{dt} = kP$, where k is a constant and t is the time in years. The population of the country doubles every twenty years.
   a. Find k.
   b. In which year will the country triple its population at the beginning of 2009?
   c. Given that at the beginning of 2009 the population was 15.1 million, what will the population be at the beginning of 2059?

3. The councils in two towns A and B have found that their populations are given by the equations $P_A = 2000e^{-0.02t}$ and $P_B = 1000e^{0.03t}$, where t is the number of years that have elapsed since January 1, 2009.
   a. Write down the annual growth rate for town B.
   b. Calculate the rate at which the population of B will be increasing at the start of 2024.
   c. During which year will the population of B surpass the population of A?

4. A vessel filled with liquid is being emptied and the volume $V\text{m}^3$ remaining after t minutes is given by the equation $V = V_0e^{-kt}$.
   a. Show that $\frac{dV}{dt} = -kV$.
   b. If one quarter of the vessel is emptied in the first 5 minutes, what fraction will remain after 10 minutes have elapsed?
   c. At what rate is the liquid flowing out:
      i. ...after 10 minutes?
      ii. ...when one quarter of the vessel is empty?

» Answers

1. A population $N(t)$ varies with time according to the equation $N(t) = Ce^{kt}$.

   a. **Show that if $\{\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^+ : \boldsymbol{a} + \boldsymbol{b} = \mathbf{1}\}$, then $\boldsymbol{N}(\boldsymbol{at} + \boldsymbol{bu}) = [\boldsymbol{N}(\boldsymbol{t})]^{\boldsymbol{a}} \times [\boldsymbol{N}(\boldsymbol{u})]^{\boldsymbol{b}}$ for $\{\boldsymbol{t}, \boldsymbol{u} \in \mathbb{R}^+\}$.**

   <u>RHS</u>

   $$\begin{aligned}[N(t)]^a \times [N(u)]^b &= [Ce^{kt}]^a \times [Ce^{ku}]^b \\ &= C^a e^{kta} \times C^b e^{kub} \\ &= C^{a+b} \times e^{kta+kub} \\ &= C^1 \times e^{k(at+bu)} \\ &= Ce^{k(at+bu)} \\ &= N(at+bu) \\ &= \text{LHS } \checkmark \ \therefore \text{True.}\end{aligned}$$

   b. **Hence, or otherwise, find $\boldsymbol{N}(\mathbf{13})$, given that $\boldsymbol{N}(\mathbf{3}) = \mathbf{10}$ and $\boldsymbol{N}(\mathbf{18}) = \mathbf{100}$.**

   $N(at + bu) = [N(t)]^a \times [N(u)]^b$

   If $t = 3$ and $u = 18$:

   $$\begin{aligned}N(3a + 18b) &= [N(3)]^a \times [N(18)]^b \\ &= 10^a \times 100^b\end{aligned}$$

   $3a + 18b = 13 \quad ...(1)$

   $a + b = 1 \quad ...(2)$

   Solving simultaneously reveals that $a = \frac{1}{3}, b = \frac{2}{3}$.

   $$\begin{aligned}N(13) &= 10^{\frac{1}{3}} \times 100^{\frac{2}{3}} \\ &= 10^{\frac{1}{3}} \times 10^{\frac{4}{3}} \\ &= 10^{\frac{5}{3}}\end{aligned}$$

2. The rate of increase of a population $P(t)$ is governed by the equation $\frac{dP}{dt} = kP$, where k is a constant and t is the time in years. The population of the country doubles every twenty years.

   a. **Find $\boldsymbol{k}$.**

   $P = P_0 e^{kt}$ where P_0 is the initial population.

   When $t = 20$, $P = 2P_0$.

   $$\begin{aligned}\therefore 2P_0 &= P_0 e^{20k} \\ 2 &= e^{20k} \\ \ln 2 &= 20k \\ k &= \frac{\ln 2}{20}\end{aligned}$$

b. **In which year will the country triple its population at the beginning of 2009?**

$$\begin{aligned} 3P_0 &= P_0 e^{kt} \\ 3 &= e^{kt} \\ \ln 3 &= kt \\ t &= \frac{\ln 3}{k} \\ t &\approx 31.6992 \ldots \end{aligned}$$

∴ The population will triple its original size during 2040.

c. **Given that at the beginning of 2009 the population was 15.1 million, what will the population be at the beginning of 2059? Answer in scientific notation to three significant figures.**

$$\begin{aligned} P &= P_0 e^{kt} \\ &= (15.1 \times 10^6) e^{50k} \\ &\approx 8.54 \times 10^7 \end{aligned}$$

3. The councils in two towns A and B have found that their populations are given by the equations $P_A = 2000e^{-0.02t}$ and $P_B = 1000e^{0.03t}$, where t is the number of years that have elapsed since January 1, 2009.

a. **Write down the annual growth rate for town B.**

0.03, or 3%.

b. **Calculate the rate at which the population of B will be increasing at the start of 2024.**

$$\frac{dP_B}{dt} = 30e^{0.03t}$$

When $t = 15$,

$$\begin{aligned} \frac{dP_B}{dt} &= 30e^{0.03\times 15} \\ &\approx 47.0493 \ldots \end{aligned}$$

∴ The population will be increasing at a rate of approximately 47 people per year.

c. **During which year will the population of B surpass the population of A?**

$$\begin{aligned} P_B &> P_A \\ 1000e^{0.03t} &> 2000e^{-0.02t} \\ e^{0.03t} &> 2e^{-0.02t} \\ 0.03t &> \ln 2 - 0.02t \\ 0.05t &> \ln 2 \\ t &> 20 \times \ln 2 \\ t &> 13.8629 \ldots \end{aligned}$$

∴ Population B will surpass Population A during 2022.

4. A vessel filled with liquid is being emptied and the volume $V\text{m}^3$ remaining after t minutes is given by the equation $V = V_0 e^{-kt}$.

a. **Show that $\frac{dV}{dt} = -kV$.**

$$V = V_0 e^{-kt}$$
$$\frac{dV}{dt} = -kV_0 e^{-kt}$$
$$= -kV \text{ since } V = V_0 e^{-kt}.$$

b. **If one quarter of the vessel is emptied in the first 5 minutes, what fraction will remain after 10 minutes have elapsed?**

$$\frac{3}{4}V_0 = V_0 e^{-5k}$$
$$\frac{3}{4} = e^{-5k}$$
$$\ln\frac{3}{4} = -5k$$
$$\frac{1}{5}\ln\frac{3}{4} = -k$$
$$\therefore V = V_0 e^{\left(\frac{1}{5}\ln\frac{3}{4}t\right)}$$

When $t = 10$,

$$V = V_0 e^{\left(\frac{1}{5}\ln\frac{3}{4}\times 10\right)}$$
$$\frac{V}{V_0} = e^{\left(2\ln\frac{3}{4}\right)}$$
$$= e^{\left(\ln\frac{9}{16}\right)}$$
$$= \frac{9}{16}$$

c. At what rate is the liquid flowing out:

i. **...after 10 minutes?**

$$\frac{dV}{dt} = -kV$$
$$\frac{dV}{dt} = \frac{1}{5}\ln\frac{3}{4} \times \frac{9}{16}V_0$$
$$= \frac{9}{80}\ln\frac{3}{4}V_0$$

$\therefore$ The liquid is flowing out at a rate of $\frac{9}{80}\ln\frac{3}{4}V_0$ m^3/minute.

ii. **...when one quarter of the vessel is empty?**

$$\frac{dV}{dt} = -kV$$
$$\frac{dV}{dt} = \frac{1}{5}\ln\frac{3}{4} \times \frac{3}{4}V_0$$
$$= \frac{3}{20}\ln\frac{3}{4}V_0$$

$\therefore$ The liquid is flowing out at a rate of $\frac{3}{20}\ln\frac{3}{4}V_0$ m^3/minute.