2. \( V = 60t \) (t seconds, \( \text{Vml} \))

(a) When \( t = 3 \), \( V = 180\text{ml} \)
(b) When \( t = 0 \), \( V = 0\text{ml} \)
    i.e. initially empty
(c) When \( t = 5 \), \( V = 300\text{ml} \)
(d) \( 60\text{ml/s} \).

6. \( \frac{dE}{dt} = t + 3 \) (t minutes, \( E \text{cm}^3 \))

(a) (i) When \( t = 0 \), \( \frac{dE}{dt} = 3 \)
    \[ \therefore 3\text{cm}^3/\text{min} \]
(ii) When \( t = 10 \), \( \frac{dE}{dt} = 13 \)
    \[ \therefore 13\text{cm}^3/\text{min} \]
(b) \( E = \int (t + 3) \, dt \)
    \[ = \frac{t^2}{2} + 3t + c \]
    But when \( t = 0 \), \( E = 0 \)
    \[ \therefore c = 0 \]
    \[ \therefore E = \frac{1}{2} t^2 + 3t \]
(c) (i) When \( t = 10 \), \( E = 50 + 30 \)
    \[ = 80 \]
    \[ = 80\text{m}^3 \]
(ii) When \( t = 20 \), \( E = 200 + 60 \)
    \[ = 260 \]
    \[ \therefore \text{Between } t = 10 \text{ & } t = 20, \]
    \[ 180\text{m}^3 \text{ is moved.} \]
8. \( \frac{dp}{dt} = 12t - 3t^2 \) (t years, P wallabies)

When \( t=0 \), \( P=25 \). \( \{0 \leq t \leq 6\} \)

(a) \( P = \int (12t - 3t^2) \, dt \)

\[ = 6t^2 - t^3 + c \]

When \( t=0 \), \( P=25 \). \( \Rightarrow c=25 \)

\[ P = 25 + 6t^2 - t^3 \]

(b) Stationary points occur when \( \frac{dp}{dt} = 0 \).

i.e. \( 12t - 3t^2 = 0 \)

\[ 3t(4-t) = 0 \]

\( t = 0, 4 \)

Test stationary points:

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dp}{dt} )</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>-36</td>
</tr>
</tbody>
</table>

- Stationary point at \( t=4 \) is a maximum.

(c) When \( t=4 \), \( P = 57 \).

- Maximum population is 57 wallabies.

(d) \( \frac{d^2p}{dt^2} = 12 - 6t \)

Points of inflexion occur when \( \frac{d^2p}{dt^2} = 0 \)

i.e. \( 12 - 6t = 0 \)

\( t = 2 \)

- Population is increasing most rapidly after 2 years.
11. \( \frac{dp}{dt} = \frac{-2}{t+1} \) (t days, P perfume content)

(a) \( P = \int \frac{-2}{t+1} \, dt \)

\[ = -2 \ln (t+1) + c \]

When \( t=0 \), \( P = 6.8 \).

\[ 6.8 = -2 \ln (1) + c \]

\[ c = 6.8 \]

\[ P = -2 \ln (t+1) + 6.8 \]

(b) Perfume will run out when \( P = 0 \).

i.e. \(-2 \ln (t+1) + 6.8 = 0\)

\[ -2 \ln (t+1) = -6.8 \]

\[ \ln (t+1) = 3.4 \]

\[ t+1 = e^{3.4} \]

\[ t = e^{3.4} - 1 \]

\[ \approx 28.9641000 \ldots \]

\[ \approx 29 \text{ days (nearest day)} \]

13. \( \frac{dx}{dt} = 250(e^{-0.2t} - 1) \) (t seconds, x metres)

(a) When \( t=0 \), \( \frac{dx}{dt} = 250(e^0 - 1) \)

\[ = 0 \]

\[ \therefore 0 \text{ m/s} \]

(b) As \( t \to \infty \), \( e^{-0.2t} \to 0 \).

\[ \therefore \text{As } t \to \infty, \frac{dx}{dt} \to -250 \]

\[ \therefore \text{Eventual speed (irrespective of direction)} \]

is \( 250 \text{ m/s} \).

(c) \( x = 250 \int (e^{-0.2t} - 1) \, dt \)

\[ = 250 \left(-5e^{-0.2t} - t\right) + c \quad \text{when } t=0, \, x=200. \]

\[ 200 = -1250 + c \]

\[ c = 1450 \]

\[ x = -250(5e^{-0.2t} + t) + 1450 \]
16. (a) \( W \) is decreasing when graph is negative, i.e. \( 0 < t < 6 \).

\( W \) is increasing when graph is positive, i.e. \( t > 6 \).

(b) Minimum for \( W \) occurs at stationary point, where graph is equal to zero, i.e. \( t = 6 \).

(c) \( W \) increases most rapidly at highest point on the graph, i.e. \( t = 12 \).

(d) \( W \) stabilises, since \( \frac{dW}{dt} \to 0 \) as \( t \to \infty \).

\[ \begin{align*}
18. & \quad \frac{dV}{dt} = -2 + \frac{t}{10} \text{ m}^3/\text{s} \\
& \quad \text{(a) When } t = 0, \quad \frac{dV}{dt} = -2 \text{ m}^3/\text{s}.
& \quad \text{(b) Tap turns off when } \frac{dV}{dt} = 0, \\
& \quad \text{ i.e. } 0 = -2 + \frac{t}{10} \\
& \quad \frac{t}{10} = 2 \\
& \quad t = 20 \text{ seconds.}
& \quad \text{(c) } V = \int (-2 + \frac{t}{10}) \, dt \\
& \quad = -2t + \frac{t^2}{20} + c \\
& \quad \text{When } t = 20, \quad V = 500. \\
& \quad 500 = -40 + \frac{400}{20} + c \\
& \quad c = 520 \\
& \quad \therefore V = 520 - 2t + \frac{t^2}{20}.
& \quad \text{(d) When } t = 0, \\
& \quad V = 520. \\
& \quad \therefore 20 \text{ m}^3 \text{ of water is lost by the time } t = 20. \\
& \quad \text{(c) In order for } 320 \text{ m}^3 \text{ of water to be let out, } \\
& \quad 280 \text{ m}^3 \text{ must be let out before gradually turning off the tap.} \\
& \quad \text{When the tap is fully on, the flow rate is } -2 \text{ m}^3/\text{s}. \\
& \quad \therefore 280 \text{ m}^3 \text{ would take } \\
& \quad 140 \text{ seconds, i.e. 2 minutes } \\
& \quad \& 20 \text{ seconds.} 
\end{align*} \]
19. \( N(t) = \frac{A}{2 + e^{-t}} \) (\( t \) months, \( N \) insects)

(a) When \( t = 0 \), \( N = 3 \times 10^5 \).
\[
3 \times 10^5 = \frac{A}{2 + 1}
\]
\[
3 \times 10^5 = \frac{A}{3}
\]
\[
A = 9 \times 10^5.
\]
\[
\therefore N = 9 \times 10^5 \left(2 + e^{-t}\right)^{-1}
\]

(b) When \( t = 1 \),
\[
N = \frac{9 \times 10^5}{2 + e^{-1}}
\]
\[
= 380,086.91842\ldots
\]
\[
\approx 380,087 \text{ (nearest insect)}
\]

(c) As \( t \to \infty \), \( e^{-t} \to 0 \).
\[
\therefore \text{As} \ t \to \infty, \ N \to \frac{9 \times 10^5}{2}
\]
\[
\to 4.5 \times 10^5 \text{ insects}.
\]

(d) Rate of population increase is given by:
\[
\frac{dN}{dt} = -9 \times 10^5 \left(2 + e^{-t}\right)^{-2} (e^{-t})
\]
\[
= 9 \times 10^5 e^{-t} (2 + e^{-t})^{-2}.
\]
\[
\text{ (i.e. } \frac{9 \times 10^5 e^{-t}}{(2 + e^{-t})^2}\text{ )}
21. \[ \frac{dI}{dt} = -5 + 4 \cos \left( \frac{\pi}{12} t \right) \quad \left( t=0 \text{ is 2am, 20th October} \right) \]

(a) \[ I = \int (-5 + 4 \cos \left( \frac{\pi}{12} t \right)) \, dt \]
\[ = -5t + \frac{48}{\pi} \sin \left( \frac{\pi}{12} t \right) + C \]

When \( t = 0 \), \( I = 18,000 \).
\[ C = 18,000 \]
\[ I = 18,000 - 5t + \frac{48}{\pi} \sin \left( \frac{\pi}{12} t \right) \]

(b) \( 4 \cos \left( \frac{\pi}{12} t \right) \leq 4 \) for all values of \( t \).
\[ -5 + 4 \cos \left( \frac{\pi}{12} t \right) \leq -1. \]

\[ \frac{dI}{dt} \leq -1, \text{ i.e. I is decreasing for all values of } t, \therefore \text{ the ice is always melting.} \]

(c) 120 days = 2880 hours.

When \( t = 2880 \),
\[ I = 18,000 - 5(2,880) + \frac{48}{\pi} \sin (240\pi) \]
\[ = 18,000 - 14,400 + 0 \]
\[ = 3,600 \]

\[ \therefore \text{In 120 days time, there will be 3,600 tonnes of ice & snow left.} \]
Exercise 6F

1. Find \( y \) as a function of \( t \) if:
   \[
   \begin{align*}
   (a) & \quad \frac{dy}{dt} = 3, \text{ and } y = -1 \text{ when } t = 0, \\
   (b) & \quad \frac{dy}{dt} = 1 - 2t, \text{ and } y = 2 \text{ when } t = 0, \\
   (c) & \quad \frac{dy}{dt} = \cos t, \text{ and } y = 1 \text{ when } t = 0, \\
   (d) & \quad \frac{dy}{dt} = e^t, \text{ and } y = 0 \text{ when } t = 0.
   \end{align*}
   \]

2. Orange juice is being poured into a glass. After \( t \) seconds there are \( V \) ml of juice in the glass, where \( V = 60t \).
   (a) How much juice is in the glass after 3 seconds?
   (b) Show that the glass was empty to begin with.
   (c) If the glass takes 5 seconds to fill, what is its capacity?
   (d) At what rate is the glass being filled?

3. Water is being pumped into a tank at the rate of \( \frac{dV}{dt} = 300 \) litres per minute, where \( V \) litres is the volume of water in the tank after \( t \) minutes of pumping. The tank had 1500 litres of water in it at time \( t = 0 \).
   (a) Show that \( V = 300t + 1500 \).
   (b) How long will the pump take to fill the tank if the tank holds 6000 litres?

4. The quantity of fuel, \( Q \) litres, in a tanker \( t \) minutes after it has started to empty is given by \( Q = 200(400 - t^3) \). Initially the tanker was full.
   (a) Find the initial quantity of fuel in the tanker.
   (b) Find the quantity of fuel in the tanker after \( 1 \) minute.
   (c) Find the time taken for the tanker to empty.
   (d) Show that \( \frac{dQ}{dt} = -400t \), and hence find the rate at which the tanker is emptying after \( 5 \) minutes.

5. Water is flowing out of a tank at the rate of \( \frac{dV}{dt} = 10t - 250 \), where \( V \) is the volume in litres remaining in the tank at time \( t \) minutes after time zero.
   (a) When does the water stop flowing?
   (b) Given that the tank still has 20 litres left in it when the water flow stops, show that the volume \( V \) at any time is given by \( V = 5t^2 - 250t + 3145 \).
   (c) How much water was initially in the tank?

6. A colony of ants is building a nest. The rate at which the ants are moving the earth is given by \( \frac{dE}{dt} = t + 3 \) cubic centimetres per minute.
   (a) At what rate are the ants moving the earth:
      (i) initially, 
      (ii) after 10 minutes?
   (b) Integrate to find \( E \) as a function of \( t \). [HINT: Find the constant of integration by assuming that when \( t = 0 \), \( E = 0 \).]
   (c) How much earth is moved by the ants in:
      (i) the first 10 minutes, 
      (ii) the next 10 minutes?
7. The share price $P$ of the Eastcom Bank $t$ years after it opened on 1st January 1970 was $P = -0.4t^2 + 4t + 2$.

(a) What was the initial share price?
(b) What was the share price after one year?
(c) At what rate was the share price increasing after two years?
(d) By letting $\frac{dP}{dt} = 0$, show that the maximum share price was $\$12$, at the start of 1975.
(e) The directors decided to close the bank when the share price fell back to its initial value. When did this happen?

8. Twenty-five wallabies are released on Wombat Island and the population is observed over the next six years. It is found that the rate of increase in the wallaby population is given by $\frac{dP}{dt} = 12t - 3t^3$, where time $t$ is measured in years.

(a) Show that $P = 25 + 6t^2 - t^3$.
(b) After how many years does the population reach a maximum? [HINT: Let $\frac{dP}{dt} = 0$.]
(c) What is the maximum population?
(d) When does the population increase most rapidly? [HINT: Let $\frac{d^2P}{dt^2} = 0$.]

9. For a certain brand of medicine, the amount $M$ present in the blood after $t$ hours is given by $M = 3t^2 - t^3$, for $0 \leq t \leq 3$.

(a) Sketch a graph of $M$ against $t$, showing any stationary points and points of inflexion.
(b) When is the amount of medicine in the blood a maximum?
(c) When is the amount of medicine increasing most rapidly?

10. When a jet engine starts operating, the rate of fuel burn, $R$ kg per minute, $t$ minutes after startup is given by $R = 10 + \frac{10}{1 + 2t}$.

(a) What is the rate of fuel burn after:
   (i) 2 minutes,
   (ii) 7 minutes?
(b) What limiting value does $R$ approach as $t$ increases?
(c) Draw a sketch of $R$ as a function of $t$.
(d) Show that approximately 83.5 kg of fuel is burned in the first 7 minutes.

11. The rate at which a perfume ball loses its scent over time is $\frac{dP}{dt} = -\frac{2}{t + 1}$, where $t$ is measured in days.

(a) Find $P$ as a function of $t$ if the initial perfume content is 6.8.
(b) How long will it be before the perfume in the ball has run out and it needs to be replaced? (Answer correct to the nearest day.)

12. A certain brand of medicine tablet is in the shape of a sphere with diameter 5 mm. The rate at which the pill dissolves is $\frac{dr}{dt} = -k$, where $r$ is the radius of the sphere at time $t$ hours, and $k$ is a positive constant.

(a) Show that $r = \frac{5}{2} - kt$.
(b) If the pill dissolves completely in 12 hours, find $k$.

13. A ball is falling through the air and experiences air resistance. Its velocity, in metres per second at time $t$, is given by $\frac{dx}{dt} = 250(e^{-0.2t} - 1)$, where $x$ is the height above the ground.

(a) What is its initial speed?
(b) What is its eventual speed?
(c) Find $x$ as a function of $t$, if the ball is initially 200 metres above the ground.
14. The graph shows the level of pollution in a river between 1995 and 2000. In 1995, the local council implemented a scheme to reduce the level of pollution in the lake. Comment briefly on whether this scheme worked and how the level of pollution changed. Include mention of the rate of change.

15. The graph to the right shows the share price \( P \) in Penn & Penn Stationery Suppliers \( t \) months after 1st January.
   (a) When is the price maximum and when is it minimum?
   (b) When is the price increasing and when is it decreasing?
   (c) When is the share price increasing most rapidly?
   (d) When is the share price increasing at an increasing rate?
   (e) Sketch a possible graph of \( \frac{dP}{dt} \) as a function of time \( t \).

16. The graph to the right shows the rate \( \frac{dW}{dt} \) at which the average weight \( W \) of bullocks at St Vidgeon station was changing \( t \) months after a drought was officially proclaimed.
   (a) When was the average weight decreasing and when was it increasing?
   (b) When was the average weight at a minimum?
   (c) When was the average weight increasing most rapidly?
   (d) What appears to have happened to the average weight as time went on?
   (e) Sketch a possible graph of the average weight \( W \).

17. The number \( U \) of unemployed people at time \( t \) was studied over a period of time. At the start of this period, the number of unemployed was 690000.
   (a) Throughout the study, \( \frac{dU}{dt} > 0 \). What can be deduced about \( U \) over this period?
   (b) The study also found that \( \frac{d^2U}{dt^2} < 0 \). What does this indicate about the changing unemployment level?
   (c) Sketch a graph of \( U \) against \( t \), showing this information.

18. A tap on a large tank is gradually turned off so as not to create any hydraulic shock. As a consequence, the flow rate while the tap is being turned off is given by \( \frac{dV}{dt} = -2 + \frac{1}{10}t \) m³/s.
   (a) What is the initial flow rate, when the tap is fully on?
   (b) How long does it take to turn the tap off?
   (c) Given that when the tap has been turned off there are still 500 m³ of water left in the tank, find \( V \) as a function of \( t \).
   (d) Hence find how much water is released during the time it takes to turn the tap off.
   (e) Suppose that it is necessary to let out a total of 300 m³ from the tank. How long should the tap be left fully on before gradually turning it off?

19. A scientist studying an insect colony estimates the number \( N(t) \) of insects after \( t \) months to be \( N(t) = \frac{A}{2 + e^{-t}} \).
   (a) When the scientist begins measuring, the number of insects in the colony is estimated to be \( 3 \times 10^6 \). Find \( A \).
   (b) What is the population of the colony one month later?
   (c) How many insects would you expect to find in the nest after a long time?
   (d) Find an expression for the rate at which the population increases with time.
20. James had a full drink bottle containing 500 ml of Gatorade™. He drank from it so that the volume $V$ ml of Gatorade™ in the bottle changed at a rate given by $\frac{dV}{dt} = (\frac{3}{2}t - 20)$ ml/s.
   (a) Find a formula for $V$.
   (b) Show that it took James 50 seconds to drink the contents of the bottle.
   (c) How long, correct to the nearest second, did it take James to drink half the contents of the bottle?

21. Over spring and summer, the snow and ice on White Mountain is melting with the time of day according to $\frac{dI}{dt} = -5 + 4 \cos \frac{\pi}{12}t$, where $I$ is the tonnage of ice on the mountain at time $t$ in hours since 2:00 am on 20th October.
   (a) It was estimated at that time that there was still 18 000 tonnes of snow and ice on the mountain. Find $I$ as a function of $t$.
   (b) Explain, from the given rate, why the ice is always melting.
   (c) The beginning of the next snow season is expected to be four months away (120 days). Show that there will still be snow left on the mountain then.

Exercise 6F (Page 265)

1(a) $y = 3t - 1$  
(b) $y = 2 + t - t^2$  
(c) $y = \sin t + 1$

2(a) 180 ml  
(b) When $t = 0$, $V = 0$.  
(c) 300 ml

3(a) 60 ml/s

3(b) 15 min

4(a) 80 000 litres  
(b) 35 000 litres  
(c) 20 min

5(a) 2000 litres/min

5(b) 25 minutes  
(c) 3145 litres

6(a) 3 cm$^3$/min  
(b) 13 cm$^3$/min  
(c) $E = \frac{1}{2}t^2 + 3t$

7(a) $\$2$  
(b) $\$5-60$  
(c) $\$2-40 per annum

8(a) $t = 4$  
(b) $t = 2$  
(c) $t = 2$

9(a) $t = 2$  
(b) $t = 1$

10(a) $12$ kg/min  
(b) $10\frac{3}{4}$ kg/min

11(a) $P = 6.8 - 2 \log(t + 1)$

12(b) $k = \frac{5}{24}$

13(a) $x = 1450 - 250(5e^{-0.2t} + t)$

14 The scheme appears to have worked initially and the level of pollution decreased, but the rate at which the pollution decreased gradually slowed down and was almost zero in 2000. A new scheme would have been required to remove the remaining pollution.

15(a) It is at a maximum on 1st July and at a minimum on 1st March.

16(a) From the graph, it is clear that the function is increasing from 1st March to 1st July.

17(a) Unemployment was increasing.

18(a) $-2$ m$^3$/s  
(b) 20 s  
(c) $V = 520 - 2t + \frac{1}{20}t^2$

19(a) $A = 9 \times 10^5$  
(b) $N(1) = 380 087$  
(c) When $t$ is large, $N$ is close to $4.5 \times 10^5$.  
(d) $N' = \frac{2 \times 10^5 e^{-t}}{(4 + e^{-t})^2}$

20(a) $V = \frac{1}{2}t^2 - 20t + 500$  
(b) $t = 50 + 25\sqrt{2} \approx 15$ seconds. Discard the other answer $t = 50 - 25\sqrt{2}$ because after 50 seconds the bottle is empty.

21(a) $I = 18000 - 5t + \frac{48}{\pi} \sin \frac{\pi}{12} t$  
(b) $\frac{dI}{dt}$ has a maximum of $-1$, so it is always negative.  
(c) There will be 3800 tonnes left.