# » Differential Calculus | **Differentiating 'Unnatural' Exponentials**

Everyone knows that $\frac{d}{dx}e^x = e^x$. It's the easiest derivative ever. But what about $\frac{d}{dx}3^x$? Or $\frac{d}{dx}10^x$? What do we do when the base of the exponential is not the 'natural' number $e$, but some random constant?

Okay, so they're not *really* called 'unnatural exponentials' – but they do cause problems for us. The normal methods we have for differentiating functions don't work here; we know what to do when $x$ is the base and a constant is the power, but not vice versa. It's not a trigonometric function that differentiates onto another trigonometric function. So what do we do? Well, bright people will work out that there's a tool in our arsenal that can help us work out the derivative of *any* function. Let's give it a spin.

### » First principles

Though it can be a pain to run through and understand initially, *first principles* is useful because, as the name suggests, it's the most basic method we have for working out the derivative of any function. Here's how it plays out when we let $f(x) = 3^x$:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{by definition of first principles}\\
&= \lim_{h\to 0}\frac{3^{x+h}-3^x}{h} &&\\
&= \lim_{h\to 0}\frac{3^x3^h-3^x}{h} && \text{by index laws}\\
&= \lim_{h\to 0}3^x\left(\frac{3^h-1}{h}\right) && \text{removing a factor of } 3^x\\
&= 3^x \times \lim_{h\to 0}\frac{3^h-1}{h} && \text{removing } 3^x \text{ from the limit, since it is independent of } h
\end{aligned}$$

At this point, we're a little stuck. There doesn't seem to be much further we can do to simplify this. However, there is a small trick we can pull out to improve the situation slightly.

$$\begin{aligned}
3^x \times \lim_{h\to 0}\frac{3^h-1}{h} &= 3^x \times \lim_{h\to 0}\frac{3^h-3^0}{h} && \text{substituting } 3^0 = 1\\
&= 3^x \times \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} && \text{read it carefully!}\\
&= 3^x f'(0) && \text{by definition of first principles}
\end{aligned}$$

Without any further help, that's as far as we can go with first principles. What this tells us is that the derivative of $3^x$ is **the function itself** multiplied by its **gradient at $x = 0$.** That's nice, but... it seems to lead us into a circular argument. We can't work out the derivative function without knowing its gradient at a certain point, but we can't work out the gradient at *any* point without knowing its derivative function.

What we can see from our result, though, is that $\frac{d}{dx}3^x \neq 3^x$. This means that we can't simply say all exponential functions have derivatives just like $e^x$. So what do we do with these annoying little functions? To move forward in our effort to work out the derivatives of *any* exponential function (not just where $e$ is the base), we need to take a few steps backward first.

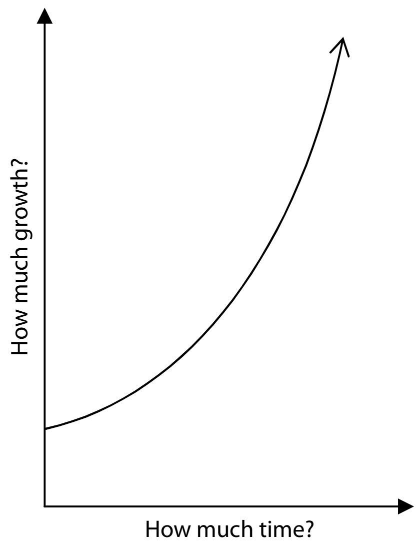
**» A scenic detour**

* If you're only after the nuts-and-bolt maths and aren't interested in taking the scenic detour, just skip to the next asterisk.

Those 'steps backward' mean spending some time thinking about logarithms. Why logarithms? To understand the reason why the longest way round can be the shortest way home, you need to remember that by definition:

$$\text{If } G = e^t, \text{then } t = \ln G.$$

Now, think for a second about how the exponential function behaves and what it tells us. The exponential function describes a quantity or amount that is growing at an ever-increasing rate. In fact, it is growing at a rate proportional to itself: the larger the amount, the faster it grows. Lots of things behave like this in real life: interest in a bank account (bank fees notwithstanding) and bacteria populations (antibiotics notwithstanding), for instance. The exponential function tells us how much something has increased after a given duration. In other words, it tells us the *growth* ($G$) after a certain *time* ($t$).




For example, suppose you've got a miraculous bank account that's paying you 100% interest per annum. Due to compound interest, your money won't double in a year – it will actually be more than that. How much more? Well, after 5 years, your amount will have grown to be $e^5 \approx 148.4$ times more than what you started with! (Not sure how it can be such a massive number? Try looking at how steep the exponential function gets when you actually graph it.) After 10 years, you'll have $e^{10}$ times more than what you started with.

Now if that's amazing, just hold onto your hat. Here is the groundbreaking idea that blew my head apart the first time I read it. If the exponential function $G = e^t$ tells us how much growth has occurred after a certain time, the logarithm function $t = \ln G$ tells us **how much time it will take to reach a certain growth.** That's what it means for exponentials and logarithms to be inverse functions; they are two sides of the same coin.

Knowing this, it shouldn't surprise you that we turn to logarithms in order to understand exponentials (and occasionally, vice versa); insights with one will often yield insights with the other. So here's the crucial question that will eventually lead us home: what's the derivative of the natural logarithm, $\ln x$?

Before you yell out your rote-learned answer, stop. Remember the whole point of this exercise: that we don't just want a *rule* to remember the answer, we want to *see how the answer is attained* and hence *understand the reason why* the answer is what it is. Ready? Let's go.

* Let $y = \ln x$.

Then, by definition, $e^y = x$.

$$
\begin{aligned}
\frac{d}{dx}e^y &= \frac{d}{dx}x && \text{differentiate each side with respect to } x \\
\frac{dy}{dy} \times \frac{d}{dx}e^y &= \frac{d}{dx}x && \text{multiply LHS by } \frac{dy}{dy} \\
\frac{dy}{dx} \times \frac{d}{dy}e^y &= \frac{d}{dx}x && \text{re-arrange denominators on LHS} \\
\frac{dy}{dx} \times e^y &= 1 && \text{evaluate derivatives} \\
\frac{dy}{dx} &= \frac{1}{e^y} && \text{isolate } \frac{dy}{dx} \\
\boldsymbol{\frac{d}{dx}\ln x} &= \boldsymbol{\frac{1}{x}} && \text{by substitution, since } y = \ln x \text{ and } e^y = x
\end{aligned}
$$

Okay. So that was an interesting piece of algebraic juggling. Now what?

### » The original problem

We actual need that important result before we can make any progress on working out the derivative of $3^x$, $10^x$, or for that matter $a^x$ where $a$ is any real constant. Now that we do have it, though, the derivative of $a^x$ can be found through a remarkably similar process to the derivative of $\ln x$. Watch and learn:

Let $y = a^x$.

Then:

$$
\begin{aligned}
\ln y &= \ln a^x && \text{taking the log of each side} \\
&= x \ln a && \text{according to the power rule of logarithms} \\
\frac{d}{dx}\ln y &= \frac{d}{dx}x \ln a && \text{differentiate each side with respect to } x \\
\frac{dy}{dy} \times \frac{d}{dx}\ln y &= \frac{d}{dx}x \ln a && \text{looking familiar?} \\
\frac{dy}{dx} \times \frac{d}{dy}\ln y &= \frac{d}{dx}x \ln a && \text{re-arrange denominators on LHS} \\
\frac{dy}{dx} \times \frac{1}{y} &= \ln a && \text{evaluate derivatives} \\
\frac{dy}{dx} &= y \ln a && \text{isolate } \frac{dy}{dx} \\
\boldsymbol{\frac{d}{dx}a^x} &= \boldsymbol{a^x \ln a} && \text{by substitution, since } y = a^x
\end{aligned}
$$

And we're done! Incidentally, one of the first things you should notice is that this result still makes sense when we let $a = e$. $\ln e = 1$, and so that part of the derivative 'vanishes', leaving the familiar $\frac{d}{dx}e^x = e^x$.