1. Alternative first principles notation

We have already used the following notation to formally define the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This definition comes from considering the gradient (rise ÷ run) between the points $(x, f(x))$ and $(x+h, f(x+h))$ as h approaches 0. However, this is an alternative method of defining the derivative that can sometimes be more helpful:

$$f'(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This comes from considering the gradient (rise ÷ run) between the points $(c, f(c))$ and $(x, f(x))$ as x approaches c. This amounts to the same thing as the first definition, though as you will see later, it has certain features that make it easier to work with algebraically under certain circumstances.

2. Generalising the difference of powers

Prior to this year, you have already learnt the following factorisations:

Difference of squares: $a^2 - b^2 = (a-b)(a+b)$
Difference of cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Is there a pattern here we can exploit and generalise from? You can manually verify the following results:

Difference of fourth powers: $a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3)$
Difference of fifth powers: $a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

Which leads to the following general statement:

Difference of powers: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$

Despite the fact that it involves an indefinite number of terms, we can verify it by multiplying out the RHS:

$$\begin{aligned} RHS &= a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} \\ &\qquad - a^{n-1}b - a^{n-2}b^2 - \cdots - ab^{n-1} - b^n \\ &= a^n - b^n \text{ (as required)} \end{aligned}$$

We will need this to simplify our result for the *derivative of an expression involving powers of x* (or any other variable, for that matter).

3. Differentiating powers of x

In order to establish this result, we will work from first principles. Let $f(x) = x^n$.

$$\therefore f'(x) = \lim_{x \to c} \frac{x^n - c^n}{x - c}$$

$$= \lim_{x \to c} \frac{(x - c)(x^{n-1} + x^{n-2}c + x^{n-3}c^2 + \cdots + c^{n-1})}{x - c} \qquad \text{(from part 2)}$$

$$= \lim_{x \to c} (x^{n-1} + x^{n-2}c + x^{n-3}c^2 + \cdots + c^{n-1}) \qquad \text{since } x \neq c$$

$$= x^{n-1} + x^{n-1} + x^{n-1} + \cdots + x^{n-1} \qquad (n \text{ terms})$$

$$f'(x) = nx^{n-1}$$

This is an incredibly useful (and time-saving!) result. Note that it works for all real numbers, not just integers or positive numbers.