» Chain Rule

The chain rule is a powerful tool for differentiating a wide variety of functions.

$$(\mathrm{f} \circ \mathrm{g})'(x) = \mathrm{f}'[\mathrm{g}(x)]\mathrm{g}'(x)$$

Chain rule is the only efficient method for differentiating functions within functions. It's important to distinguish between the 'inside' and 'outside' functions – we must differentiate outside first, then inside.

» Instances

This generalised chain rule plays out differently depending on the functions that represent f(x) and g(x).

Chain rule for **raising a function to a power** $\frac{d}{dx}[\mathrm{f}(x)]^a = a\mathrm{f}'(x)[\mathrm{f}(x)]^{a-1}$ e.g. $\frac{d}{dx}\sin^5 x = 5\cos x\sin^4 x$	Chain rule for **sine (sin)** $\frac{d}{dx}\sin[\mathrm{f}(x)] = \mathrm{f}'(x)\cos[\mathrm{f}(x)]$ e.g. $\frac{d}{dx}\sin(e^x) = e^x\cos(e^x)$
Chain rule for **exponentials** $\frac{d}{dx}e^{\mathrm{f}(x)} = \mathrm{f}'(x)e^{\mathrm{f}(x)}$ e.g. $\frac{d}{dx}e^{(x^3)} = 3x^2e^{(x^3)}$	**Chain rule for cosine (cos)** $\frac{d}{dx}\cos[\mathrm{f}(x)] = -\mathrm{f}'(x)\sin[\mathrm{f}(x)]$ e.g. $\frac{d}{dx}\cos\left(\frac{x}{2} - 4\right) = -\frac{1}{2}\sin(\frac{x}{2} - 4)$
Chain rule for **logarithms** $\frac{d}{dx}\ln[\mathrm{f}(x)] = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ e.g. $\frac{d}{dx}\ln(\sin x) = \cot x$	**Chain rule for tangent (tan)** $\frac{d}{dx}\tan[\mathrm{f}(x)] = \mathrm{f}'(x)\sec^2[\mathrm{f}(x)]$ e.g. $\frac{d}{dx}\tan(\ln x) = \frac{1}{x}\sec^2(\ln x)$

» Exercises

1. Differentiate the following:

a. $\cos(3x + 4)$
b. $\cos(\ln x)$
c. $\sin(\cos x)$
d. $\tan(\sqrt{x})$
e. $\ln(\sqrt{x})$
f. $\ln(x^3 + e^x)$
g. $(\ln x)^3$
h. $e^{\sin x}$
i. $e^{\ln x}$
j. $\ln(\tan 3x)$

2. More challenging:

a. $\ln(5x + \sin x)$
b. $e^{3\tan(-x)}$
c. $\tan(\tan x - \frac{2}{x})$
d. $\sin^3(\frac{x^2}{3})$
e. $\sqrt[5]{\ln x}$
f. $e^{(x^2\ln x)}$
g. $\cos(-4\cos x)$
h. $\sqrt{\frac{\sin x}{\sin x+1}}$
i. $\tan(x^2\sin x)$
j. $\tan^7(e^{4x})$

Answers

Question 1

a. $\frac{d}{dx}\cos(3x+4) = -3\sin(3x+4)$

b. $\frac{d}{dx}\cos(\ln x) = -\frac{1}{x}\sin(\ln x)$

c. $\frac{d}{dx}\sin(\cos x) = -\sin x\cos(\cos x)$

d. $\frac{d}{dx}\tan(\sqrt{x}) = \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}$

e. $\frac{d}{dx}\ln(\sqrt{x}) = \frac{1}{2x}$

f. $\frac{d}{dx}\ln(x^3+e^x) = \frac{3x^2+e^x}{x^3+e^x}$

g. $\frac{d}{dx}(\ln x)^3 = \frac{3}{x}(\ln x)^2$

h. $\frac{d}{dx}e^{\sin x} = \cos x e^{\sin x}$

i. $\frac{d}{dx}e^{\ln x} = \frac{d}{dx}x = 1$

j. $\frac{d}{dx}\ln(\tan 3x) = 3\mathrm{cosec}(3x)\sec(3x)$

Question 2

a. $\frac{d}{dx}\ln(5x+\sin x) = \frac{5+\cos x}{5x+\sin x}$

b. $\frac{d}{dx}e^{3\tan(-x)} = -3\sec^2 x\, e^{3\tan(-x)}$

c. $\frac{d}{dx}\tan(\tan x - \frac{2}{x}) = (\sec^2 x + \frac{2}{x^2})\sec^2\left(\tan x - \frac{2}{x}\right)$

d. $\frac{d}{dx}\sin^3(\frac{x^2}{3}) = 2x\sin^2(\frac{x^2}{3})\cos(\frac{x^2}{3})$

e. $\frac{d}{dx}\sqrt[5]{\ln x} = \frac{1}{5x(\ln x)^{\frac{4}{5}}}$

f. $\frac{d}{dx}e^{(x^2\ln x)} = (2\ln x + 1)xe^{(x^2\ln x)}$

g. $\frac{d}{dx}\cos(-4\cos x) = 4\sin x\sin(4\cos x)$

h. $\frac{d}{dx}\sqrt{\frac{\sin x}{\sin x+1}} = \frac{\cos x}{2\sqrt{\sin x(\sin x+1)^3}}$

i. $\frac{d}{dx}\tan(x^2\sin x) = (2\sin x + x\cos x)x\sec^2(x^2\sin x)$

j. $\frac{d}{dx}\tan^7(e^{4x}) = 28\sec^2(e^{4x})\tan^6(e^{4x})$