» Cambridge Exercise 10E: Q20

Use the definition of the derivative to show that the derivative of an even function is odd.

$f(x)$ is a even function, i.e. $f(x) = f(-x)$.

Aim: show that $f'(x)$ is an odd function, i.e. $f'(-x) = -f'(x)$.

By definition (according to first principles), we can express the derivative $f'(x)$ as:

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\therefore f'(-x) = \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h}$

$= \lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$

Conversely, $-f'(x) = \lim_{h\to 0} \frac{-f(x+h)+f(x)}{h}$

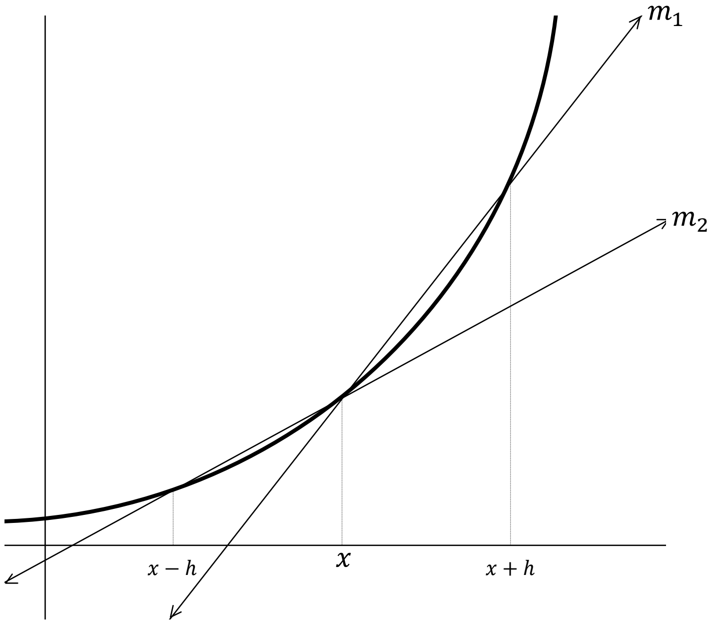
$= \lim_{h\to 0} \frac{f(x)-f(x+h)}{h}$

But as $h \to 0$, $\frac{f(x-h)-f(x)}{h} \to \frac{f(x)-f(x+h)}{h}$ (see illustration).

$\therefore -f'(x) = \lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$

$= f'(-x)$

Therefore, $f'(x)$ is an odd function.



As the distance h decreases, the gradients of the two lines m_1 and m_2 approach the same value. The value they approach (from 'above' and 'below', so to speak) is the true gradient of $f(x)$ at the point x, which we write as $f'(x)$.

The gradient of m_1 is $\frac{f(x)-f(x+h)}{h}$.

The gradient of m_2 is $\frac{f(x-h)-f(x)}{h}$.

$\therefore$ As $h \to 0, \frac{f(x-h)-f(x)}{h} \to \frac{f(x)-f(x+h)}{h}$.