» Prelim Final Exam – Solutions

In class we went through Q2, Q3 and Q4b. Here are solutions for the rest of the questions that we did not cover.

» Q1

a) Write down the exact value of:

i) $\sin 150 = \sin 30$

$= \frac{1}{2}$

ii) $\operatorname{cosec} 315 = \frac{1}{\sin 315}$

$= \frac{-1}{\sin 45}$

$= -\sqrt{2}$

b) Find the exact value of x if:

$$3\log_{10}4 - 2\log_{10}x = 6$$

$$\log_{10}(4^3) - \log_{10}(x^2) = 6$$

$$\log_{10}\left(\frac{4^3}{x^2}\right) = 6$$

$$10^6 = \frac{4^3}{x^2}$$

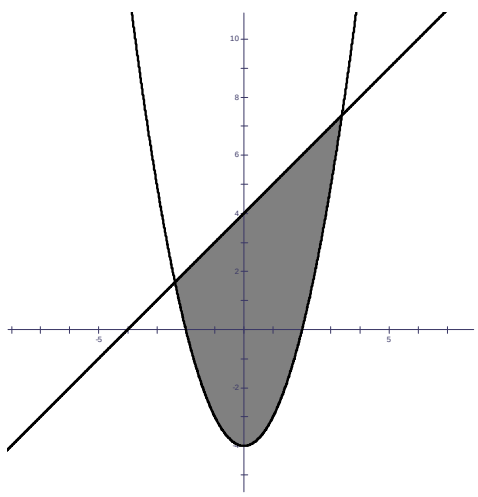
$$x^2 = \frac{4^3}{10^6}$$

$$x^2 = \frac{2^6}{10^6}$$

$$x = \frac{2^3}{10^3}$$

$$x = \frac{1}{125}$$

c) Shade the regions in the Cartesian plane which represent the simultaneous solution of $y \leq x + 4$ and $y \geq x^2 - 4$.



d) Given that $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ can be written in the form $p + q\sqrt{2}$, find the values of p and q.

$\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}-1}$

$= \frac{(\sqrt{2}+1)^2}{2-1}$

$= 2 + 2\sqrt{2} + 1$

$= 3 + 2\sqrt{2}$

$\therefore p = 3;\ q = 2.$

e) For what values of k is the expression $kx^2 - (3k - 1)x + k$ positive definite?

An expression is *positive definite* if it always remains above the x-axis – that is, if its value is always positive. Since it is a quadratic function, this would mean it has no roots (if there were roots, that would mean parts of the function would be underneath the x-axis and thus negative).

Additionally, the leading co-efficient would need to be positive, otherwise the entire function would be beneath the x-axis.

Therefore we can state the following:

$\boldsymbol{b^2 - 4ac < 0}$ (discriminant less than zero, meaning that there are no roots)

$\boldsymbol{k > 0}$ (leading co-efficient greater than zero, so the graph is concave up)

Since $b^2 - 4ac < 0$,

$$(-3k + 1)^2 - (4 \times k \times k) < 0$$

$$9k^2 - 6k + 1 - 4k^2 < 0$$

$$5k^2 - 6k + 1 < 0$$

$$(5k - 1)(k - 1) < 0$$

$\therefore$ For the expression to be positive definite, k must take the values $\frac{1}{5} < k < 1$.

(f) I am going to leave out this question because the answer is quite long – and most of us got it correct.

» Q4a

The diagram shows the first quadrant of the circle $x^2 + y^2 = 1$. The point A has coordinates $\left(\frac{1}{2}, 0\right)$ & B has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

i) What is the exact value of $\angle COB$?

Let $\angle AOB = \theta$.

$$\tan\theta = \frac{\sqrt{3}}{2} \div \frac{1}{2}$$
$$= \sqrt{3}$$
$$\theta = \tan^{-1}(\sqrt{3})$$
$$\theta = \frac{\pi}{3}$$

But $\angle COB = \frac{\pi}{2} - \theta$.

$$\therefore \angle COB = \frac{\pi}{2} - \frac{\pi}{3}$$
$$= \frac{\pi}{6}$$

ii) Show that the exact value of the shaded area OABC is $\frac{2\pi+3\sqrt{3}}{24}$.

$$\text{Total } A = A_{Sector\ COB} + A_{\Delta AOB}$$
$$= \frac{1}{2}r^2\theta + \frac{1}{2}bh$$
$$= \left[\frac{1}{2} \times (1)^2 \times \frac{\pi}{6}\right] + \left[\frac{1}{2} \times \frac{1}{2} \times \frac{\sqrt{3}}{2}\right]$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$
$$= \frac{2\pi+3\sqrt{3}}{24} \text{ units}^2$$

» Q4c

The point $P\left(n, \frac{n^2}{2}\right)$ lies on the parabola with equation $2y = x^2$. $A(4, 1)$ is a fixed point.

i) Show that $PA = \left(\frac{n^4}{4} - 8n + 17\right)^{\frac{1}{2}}$.

$$PA = \sqrt{(4-n)^2 + \left(1 - \frac{n^2}{2}\right)^2}$$
$$= \sqrt{16 - 8n + n^2 + 1 - n^2 + \frac{n^4}{4}}$$
$$= \sqrt{\frac{n^4}{4} - 8n + 17}$$
$$= \left(\frac{n^4}{4} - 8n + 17\right)^{\frac{1}{2}}$$

ii) As P moves, find the minimum distance of P from A.

To find the minimum value of PA, we must differentiate PA with respect to n, and then locate its turning points.

i.e. $\frac{d(PA)}{dn} = 0$

$$0 = \frac{1}{2}\left(\frac{n^4}{4} - 8n + 17\right)^{\frac{-1}{2}} \times (n^3 - 8)$$
$$= \frac{(n^3 - 8)}{2\sqrt{\frac{n^4}{4} - 8n + 17}}$$

A fraction is only equal to 0 when its numerator is equal to 0. If the numerator is 0, then the value of the denominator is inconsequential.

$$\therefore n^3 - 8 = 0$$
$$n^3 = 8$$
$$n = 2$$

We can test the nature of this turning point and deduce that it is a local minimum. Since there is only one turning point, it is also the absolute minimum.

When $n = 2$,

$$PA = \sqrt{\frac{2^4}{4} - 8(2) + 17}$$
$$= \sqrt{5} \text{ units}$$

iii) Prove that for this position of P, AP is normal to the parabola.

If AP is normal to the parabola, then:

$m_{AP} \times \frac{dy}{dx} = -1$ when $n = 2$.

When $n = 2$, P has coordinates (2,2).

$$m_{AP} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2-1}{2-4}$$
$$= \frac{-1}{2}$$

Since $2y = x^2$,

$$\mathrm{y} = \frac{1}{2}x^2$$

$\therefore \frac{dy}{dx} = x$.

Thus, when $x = 2$, $\frac{dy}{dx} = 2$.

$$\therefore m_{AP} \times \frac{dy}{dx} = \frac{-1}{2} \times 2$$
$$= -1.$$

$\therefore$ When $n = 2$, AP is normal to the parabola.