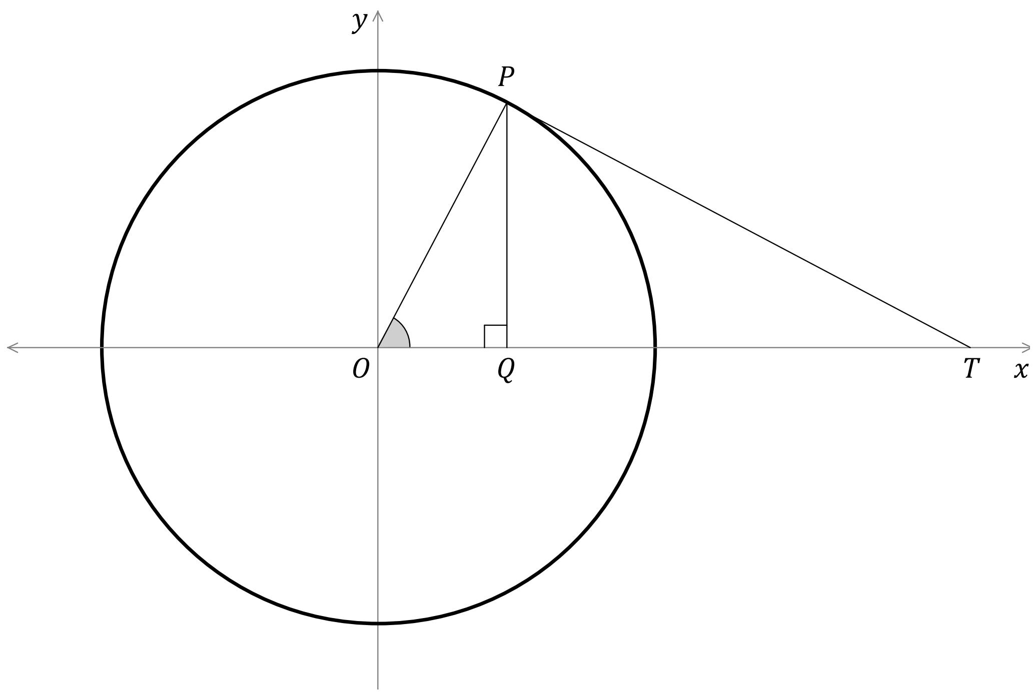
# » Trigonometric Functions on the Unit Circle

Our current definition of the trigonometric functions is in terms of the ratios of sides in right-angled triangles (we use the mnemonic SOHCAHTOA to remember which functions correspond to the ratios of which sides). With a little bit of thought, we can see these same functions in a different (but related) light that yields some very interesting observations.

Let's begin by drawing a unit circle (centre at the origin, radius 1 unit), with a point $P$ on its circumference. Point $Q$ is placed such that $PQ$ is perpendicular to the $x$-axis. Interval $PT$ is tangent to the circle. Let $\angle POQ = \theta°$.




## » Questions

1. What is the length of $OP$?
2. Express $\sin\theta$ & $\cos\theta$ in terms of the sides of $\Delta POQ$.
3. Thus, find the lengths of $PQ$ & $OQ$. Re-write the co-ordinates of $P$ in terms of these lengths.
4. Calculate the gradient of interval $OP$.
5. Use these results to write new definitions for sin, cos and tan in terms of their relationship to the unit circle.
6. What you calculated in question 4 does not seem to make sense of why "tan" is short for *tangent function*. Complete the following questions to see why this is so, and thus re-formulate your answer to question 5.
    a. What is the size of $\angle OPT$? Why is this so?
    b. Knowing the gradient of $OP$, calculate the gradient of $PT$.
    c. What is the equation of the line that passes through $P$ & $T$?
    d. Use this equation to find the co-ordinates of $T$.
    e. What is the length of interval $PT$?

## » Solutions

**1. What is the length of $OP$?**

$OP = 1$ unit, since it is a unit circle.

**2. Express $\sin\theta$ & $\cos\theta$ in terms of the sides of $\Delta POQ$.**

$\sin\theta = \frac{PQ}{OP}, \cos\theta = \frac{OQ}{OP}$.

**3. Thus, find the lengths of $PQ$ & $OQ$. Re-write the co-ordinates of $P$ in terms of these lengths.**

From the previous two questions, we can see that $PQ = \sin\theta$ and $OQ = \cos\theta$. Thus, the required co-ordinates are $P(\cos\theta, \sin\theta)$.

**4. Calculate the gradient of interval $OP$.**

$$m_{OP} = \frac{\sin\theta}{\cos\theta} \quad \left(\text{i. e. } \frac{\text{rise}}{\text{run}}\right)$$
$$= \tan\theta$$

**5. Write new definitions for sin, cos and tan in terms of their relationship to the unit circle.**

Given an angle $\theta$ measured upwards from positive $x$-axis that creates a point $P$ on the unit circle:

- $\sin\theta$ is the $y$-coordinate of $P$
- $\cos\theta$ is the $x$-coordinate of $P$
- $\tan\theta$ is the gradient of the radius to $P$

**6. What you calculated in Q4 does not seem to make sense of why "tan" is short for *tangent function*. Complete the following questions to see why this is so, and thus re-formulate your answer to Q5.**

**a. What is the size of $\angle OPT$? Why is this so?**

$\angle OPT = 90°$ (tangent is perpendicular to the radius of a circle at the point of contact)

**b. Knowing the gradient of $OP$, calculate the gradient of $PT$.**

$m_{OP} = \frac{\sin\theta}{\cos\theta}$. But $OP \perp PT$ (shown above).

$$\therefore m_{PT} = \frac{-\cos\theta}{\sin\theta}$$

**c. What is the equation of the line that passes through $P$ & $T$?**

Using point-gradient formula, we can work out the equation of $PT$ to be:

$$y - \sin\theta = \frac{-\cos\theta}{\sin\theta}(x - \cos\theta)$$

**d. Use this equation to find the co-ordinates of $T$.**

$T$ lies on the $x$-axis. Therefore, we ought to substitute $y = 0$ into the equation of $PT$ to find the $x$-coordinate of $T$.

$$0 - \sin\theta = \frac{-\cos\theta}{\sin\theta}(x - \cos\theta)$$

$$\frac{\sin^2\theta}{\cos\theta} = x - \cos\theta$$

$$x = \cos\theta + \frac{\sin^2\theta}{\cos\theta}$$

$$x = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$$

$$x = \frac{1}{\cos\theta}$$

$\therefore$ The required co-ordinates are $T(\frac{1}{\cos\theta}, 0)$.

**e. What is the length of interval $PT$?**

To calculate the distance between $P(\cos\theta, \sin\theta)$ and $T(\frac{1}{\cos\theta}, 0)$, we simply need to input their respective co-ordinates into the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$$d_{PT} = \sqrt{\left(\cos\theta - \frac{1}{\cos\theta}\right)^2 + (\sin\theta - 0)^2}$$

$$= \sqrt{\cos^2\theta - 2 + \frac{1}{\cos^2\theta} + \sin^2\theta}$$

$$= \sqrt{1 - 2 + \frac{1}{\cos^2\theta}}$$

$$= \sqrt{\frac{1}{\cos^2\theta} - 1}$$

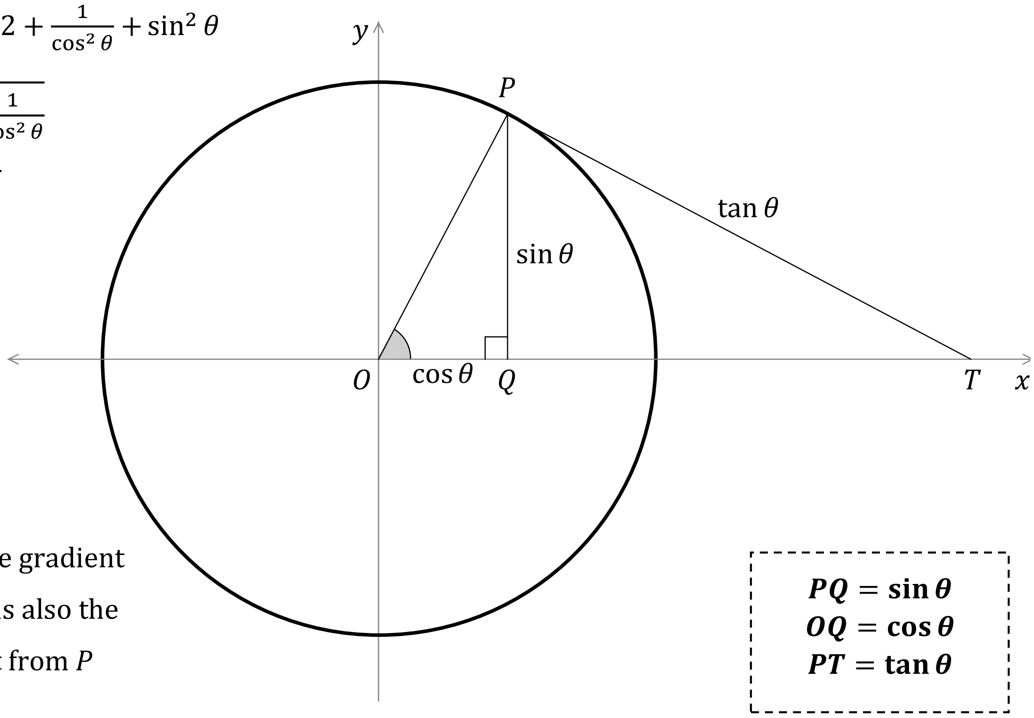
$$= \sqrt{\frac{1 - \cos^2\theta}{\cos^2\theta}}$$

$$= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \sqrt{\tan^2\theta}$$

$$= \tan\theta$$

$\therefore$ $\tan\theta$ is not only the gradient of the radius to $P$, it is also the length of the tangent from $P$ to the $x$-axis.




$PQ = \sin\theta$
$OQ = \cos\theta$
$PT = \tan\theta$