» Quadratic Theory | **Sum & Product of Roots**

Mathematics is all about patterns: identifying them, understanding them, manipulating them and learning their ramifications. Sometimes patterns appear in places you wouldn't expect them, such as the appearance of Fibonacci numbers in numerous biological settings. An interesting and unusually simple pattern emerges when exploring the relationships between the roots of polynomial equations, as we will see.

» Quadratic equations

We have already seen in class, through algebraic manipulation of the quadratic formula, the results for the *sum* and *product* of the roots of a quadratic equation. A more elegant (though less obvious) approach requires the use of quadratic identities. Watch and learn:

Suppose the roots of $ax^2 + bx + c = 0$ are α and β.

$$\begin{aligned} \therefore ax^2 + bx + c &= a(x-\alpha)(x-\beta) \\ &= a(x^2 - \beta x - \alpha x + \alpha\beta) \\ ax^2 + bx + c &= a(x^2 - (\alpha+\beta)x + \alpha\beta) \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= x^2 - (\alpha+\beta)x + \alpha\beta \end{aligned}$$

By comparison of co-efficients:

$$\begin{aligned} \frac{b}{a} &= -(\alpha+\beta) \\ \alpha+\beta &= \frac{-b}{a} \end{aligned}$$

Similarly: $\alpha\beta = \frac{c}{a}$

Okay, that's nice. But so what? As we add a layer of complexity to the problem, the plot thickens.

» Cubic equations

What happens if we try the same approach on *cubic* equations instead of *quadratic* equations?

Suppose the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β and γ.

$$\begin{aligned} \therefore ax^3 + bx^2 + cx + d &= a(x-\alpha)(x-\beta)(x-\gamma) \\ x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} &= (x-\alpha)(x-\beta)(x-\gamma) \\ &= (x^2 - (\alpha+\beta)x + \alpha\beta)(x-\gamma) \\ &= x^3 - (\alpha+\beta)x^2 + \alpha\beta x - \gamma x^2 + (\alpha+\beta)\gamma x - \alpha\beta\gamma \\ x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} &= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\alpha\gamma)x - \alpha\beta\gamma \end{aligned}$$

By comparison of co-efficients:

$$\frac{b}{a} = -(\alpha + \beta + \gamma)$$

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

Similarly:

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$$

Lastly:

$$-\alpha\beta\gamma = \frac{d}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

» The pattern

While the algebra took a bit more effort to bash through, we ended up with results just as simple as for the quadratic equation. This is intriguing, especially seeing as it is disproportionately more complicated to actually find the roots of an arbitrary cubic equation (see my write-up on *Introduction to Solving Polynomials*, in the Year 9 section). The roots of a cubic equation are incredibly difficult to calculate, and yet the relationship between them is as straightforward and algebraically clear-cut as they are for a quadratic equation.

What's more, this pattern continues. If we were to consider a *quartic* equation (polynomial with degree 4) in the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ that had roots α, β, γ and δ, we could say the following:

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$$

$$\alpha\beta\gamma\delta = \frac{e}{a}$$

The pattern also applies to quintic equations (polynomials with degree 5), hexic equations (polynomials with degree 6), heptic, octic, nonic and decic equations. You get the idea.

The set of equations that describe the relationships between the roots of polynomials (as we have shown above for quadratics, cubics and quartics) are called *Viète's formulas*, after François Viète who discovered them in the sixteenth century.