As we have already seen, numerous trigonometric properties and identities come from geometry. On the basis of the original definitions of the trig functions, mathematicians have observed properties that can be expressed algebraically (e.g. \( \sin(A+B) = \sin A \cos B + \cos A \sin B \)).

An identity called the **Pythagorean Relationship** arises when considering a generic right-angled triangle, such as the one below.

1. If \( XZ = h \) units and \( \angle YXZ = \alpha^\circ \), then calculate the sizes of \( XY \) and \( YZ \) in terms of trigonometric functions of \( \alpha \).
2. Write down the equation that relates the three sides of \( \triangle XYZ \), according to Pythagoras' theorem.
3. Simplify this equation to produce the Pythagorean Relationship.
4. Use the Pythagorean Relationship to simplify the following:
   a. \( 1 + \tan^2 x \)
   b. \( 1 + \cot^2 x \)
   c. \( \frac{1}{1 - \sin A} + \frac{1}{1 + \sin A} \)
   d. \( (\cos B + \sin B)^2 + (\cos B - \sin B)^2 \)
   e. \( \sin^2 \theta (\cot^2 \theta + 1) \)
5. Prove the following:
   a. \( \tan Y + \cot Y = \sec Y \cosec Y \)
   b. \( \sin^4 \theta - \cos^4 \theta = \sin^2 \theta - \cos^2 \theta \)
   c. \( \frac{\cot B}{1 + \cot B} = \frac{1}{1 + \tan B} \)
ANSWERS & SOLUTIONS | The Pythagorean Relationship

Here are answers and solutions to the prescribed exercises.

4. Use the Pythagorean Relationship to simplify the following:
   a. \(1 + \tan^2x = \sec^2x\)
   b. \(1 + \cot^2x = \cosec^2x\)
   c. \(\frac{1}{1 - \sin A} + \frac{1}{1 + \sin A} = 2\sec^2A\)
   d. \((\cos B + \sin B)^2 + (\cos B - \sin B)^2 = 2\)
   e. \(\sin^2\theta(\cot^2\theta + 1) = 1\)

5. Prove the following:
   a. \(\tan Y + \cot Y = \sec Y\cosec Y\)
      Step 1: (LHS) expand \(\tan Y\) and \(\cot Y\)
      Step 2: combine fractions with common denominator
      Step 3: substitute Pythagorean relationship
      Step 4: express as reciprocal trig functions
   b. \(\sin^6\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta\)
      Step 1: (LHS) factorise by difference of squares
      Step 2: substitute Pythagorean relationship
   c. \(\frac{\cot B}{1 + \cot B} = \frac{1}{1 + \tan B}\)
      Step 1: (LHS) expand \(\cot B\) in numerator and denominator
      Step 2: denominator – combine fractions
      Step 3: evaluate fractions
      Step 4: divide numerator and denominator through

EXTENSION QUESTIONS | The Pythagorean Relationship

Here are some more difficult questions for practice.

6. Prove the following:
   a. \(\sqrt{\frac{\sec^2\theta - 1}{\sec^2\theta}} = \sin\theta\cos\theta\)
   b. \(\frac{\cot x - \tan x}{\sin x\cos x} = \cosec^2x - \cosec^2x\)
   c. \((\sin^2D - 1)(\cot^2D + 1) = -\cot^2D\)
   d. \(\cosec C + \cot C = \frac{\sin C}{1 - \cos C}\)
   e. \(\sec^2x + \tan^2x\sec^2x = \sec^4x\)