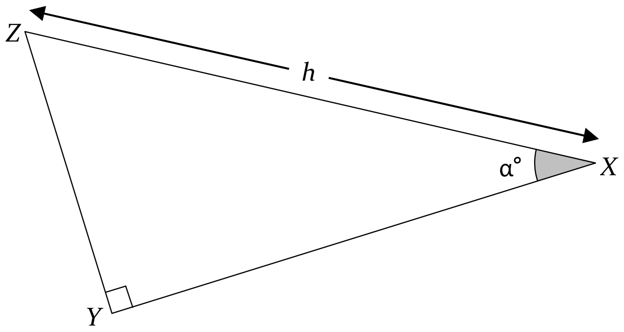
WORKSHEET | **The Pythagorean Relationship**

As we have already seen, numerous trigonometric properties and identities come from geometry. On the basis of the original definitions of the trig functions, mathematicians have observed properties that can be expressed algebraically (e.g. $\sin(A+B) = \sin A\cos B + \cos A\sin B$).

An identity called the **Pythagorean Relationship** arises when considering a generic right-angled triangle, such as the one below.



1. If $XZ = h$ units and $\angle YXZ = \alpha°$, then calculate the sizes of XY and YZ in terms of trigonometric functions of α.
2. Write down the equation that relates the three sides of ΔXYZ, according to Pythagoras' theorem.
3. Simplify this equation to produce the Pythagorean Relationship.
4. Use the Pythagorean Relationship to simplify the following:
   a. $1 + \tan^2 x$
   b. $1 + \cot^2 x$
   c. $\frac{1}{1-\sin A} + \frac{1}{1+\sin A}$
   d. $(\cos B + \sin B)^2 + (\cos B - \sin B)^2$
   e. $\sin^2\theta(\cot^2\theta + 1)$
5. Prove the following:
   a. $\tan Y + \cot Y = \sec Y \operatorname{cosec} Y$
   b. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$
   c. $\frac{\cot B}{1+\cot B} = \frac{1}{1+\tan B}$

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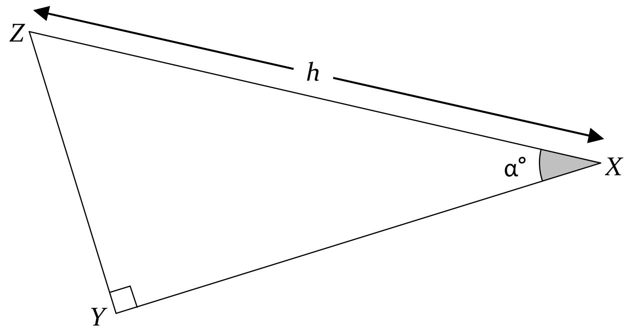

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ANSWERS & SOLUTIONS | **The Pythagorean Relationship**

Here are answers and solutions to the prescribed exercises.

4. Use the Pythagorean Relationship to simplify the following:
   a. $1+\tan^2 x = \mathbf{sec^2} \boldsymbol{x}$
   b. $1+\cot^2 x = \mathbf{cosec^2} \boldsymbol{x}$
   c. $\frac{1}{1-\sin A}+\frac{1}{1+\sin A} = \mathbf{2sec^2} \boldsymbol{A}$
   d. $(\cos B+\sin B)^2+(\cos B-\sin B)^2 = \mathbf{2}$
   e. $\sin^2\theta(\cot^2\theta+1) = \mathbf{1}$
5. Prove the following:
   a. $\tan Y+\cot Y = \sec Y \operatorname{cosec} Y$

      Step 1: (LHS) expand $\tan Y$ and $\cot Y$

      Step 2: combine fractions with common denominator

      Step 3: substitute Pythagorean relationship

      Step 4: express as reciprocal trig functions

   b. $\sin^4\theta-\cos^4\theta = \sin^2\theta-\cos^2\theta$

      Step 1: (LHS) factorise by difference of squares

      Step 2: substitute Pythagorean relationship

   c. $\frac{\cot B}{1+\cot B} = \frac{1}{1+\tan B}$

      Step 1: (LHS) expand $\cot B$ in numerator and denominator

      Step 2: denominator – combine fractions

      Step 3: evaluate fractions

      Step 4: divide numerator and denominator through

EXTENSION QUESTIONS | **The Pythagorean Relationship**

Here are some more difficult questions for practice.

6. Prove the following:
   a. $\sqrt{\frac{\sec^2\theta-1}{\sec^2\theta}} = \sin\theta\cos\theta$
   b. $\frac{\cot x-\tan x}{\sin x\cos x} = \operatorname{cosec}^2 x-\sec^2 x$
   c. $(\sin^2 D-1)(\cot^2 D+1) = -\cot^2 D$
   d. $\operatorname{cosec} C+\cot C = \frac{\sin C}{1-\cos C}$
   e. $\sec^2 x+\tan^2 x\sec^2 x = \sec^4 x$