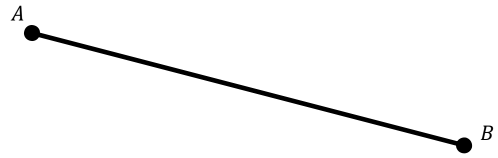
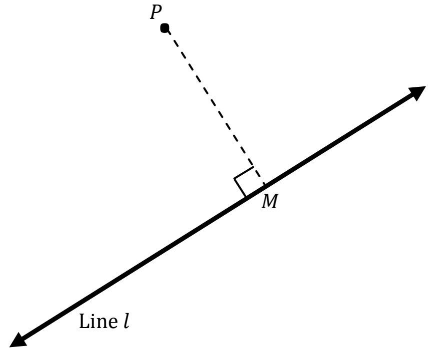
» Perpendicular Distance



When we talk about the distance between two points, say A and B, we always mean the *shortest distance* between them: that is, the length of the straight line AB that joins them.

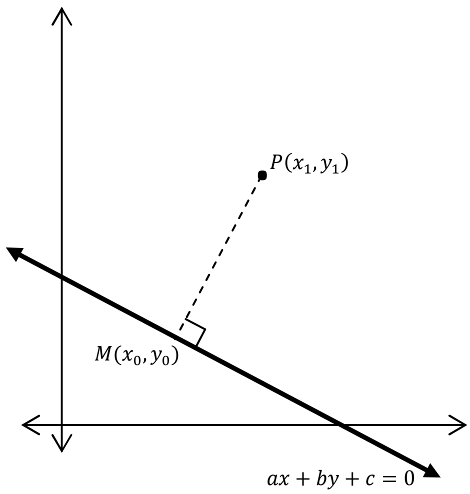


When we talk about the distance between a **point** (say P) and a **line** (say l), the same principle is at work: we are always referring to the *shortest distance* between them: that is, the length of the straight line MP where M lies on l and $MP \perp l$. This is called the *perpendicular distance* between P and l.

» Questions

1. The point Q has co-ordinates (3,4), and the line g has equation $3x + y - 5 = 0$.
    a. Sketch Q and g on a set of axes.
    b. Find the equation of line h, which is perpendicular to g and passes through Q.
    c. Find the co-ordinates of R, the point where g and h intersect.
    d. Hence calculate the perpendicular distance between Q and g.

2. The point P has co-ordinates (x_1, y_1), and the line l has equation $ax + by + c = 0$. A diagram is shown on the right.



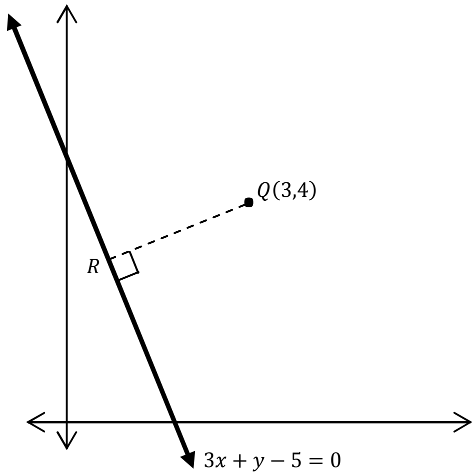
    a. Find the equation of line k, which is perpendicular to l and passes through P.
    b. Find the co-ordinates of $M(x_0, y_0)$, the point where l and k intersect.
    c. Substitute your answer to part (b) into the following expressions and simplify:
        i. $x_1 - x_0$
        ii. $y_1 - y_0$
    d. Hence calculate d_{PM} and prove that the perpendicular distance between P and l is given by the following formula:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

    e. Use this formula with the information provided in question 1, and compare the amount of working required using the previous method and the newly-derived formula.

» Answers

1. The point Q has co-ordinates (3,4), and the line g has equation $3x + y - 5 = 0$.



   a. See diagram showing Q and g on the right.

   b. Re-arranging g yields the equation $y = -3x + 5$.
   ∴ The equation of the line perpendicular to g has the form $y = \frac{x}{3} + k$. Since this line passes through Q, we can substitute its co-ordinates into the equation to determine the value of k.

$$4 = \frac{3}{3} + k$$
$$k = 3$$
$$\therefore y = \frac{x}{3} + 3 \text{ is the equation of line } h\text{, which is perpendicular to } g.$$

   c. We can now solve this equation simultaneously with g in order to find the co-ordinates of R.

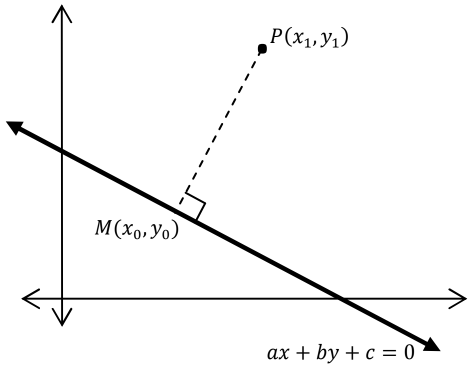
$$\frac{x}{3} + 3 = -3x + 5$$
$$x + 9 = -9x + 15$$
$$10x = 6$$
$$x = \frac{3}{5}$$

   Substituting this value into either of the lines allows us to find that $y = 3\frac{1}{5}$.
   ∴ The co-ordinates of R are $\left(\frac{3}{5}, 3\frac{1}{5}\right)$.

   d. The perpendicular distance between Q and g is synonymous with the length of interval QR.

$$d_{QR} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{\left(3 - \frac{3}{5}\right)^2 + \left(4 - 3\frac{1}{5}\right)^2}$$
$$= \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$
$$= \sqrt{\frac{144+15}{25}}$$
$$= \frac{\sqrt{160}}{5}$$
$$d = \frac{4\sqrt{10}}{5} \text{ units}$$

2. The point P has co-ordinates (x_1, y_1), and the line l has equation $ax + by + c = 0$. A diagram is shown on the right.



a. Re-arranging the equation of line l ($ax + by + c = 0$) yields the following result:

$$by = -ax - c$$
$$y = \frac{-a}{b}x - \frac{c}{b}$$

∴ Gradient of line l is $\frac{-a}{b}$. Hence, the gradient of a line perpendicular to l is $\frac{b}{a}$.

∴ Equation of line k, perpendicular to l passing through P, is:

$$y - y_1 = \frac{b}{a}(x - x_1)$$
$$y = \frac{b}{a}x - \frac{b}{a}x_1 + y_1$$

b. Let k and l intersect at point $M(x_0, y_0)$. Solve k and l simultaneously to determine x_0.

$$\frac{-a}{b}x_0 - \frac{c}{b} = \frac{b}{a}x_0 - \frac{b}{a}x_1 + y_1$$
$$-a^2x_0 - ac = b^2x_0 - b^2x_1 + aby_1 \qquad \text{[multiply both sides by } ab\text{]}$$
$$-a^2x_0 - b^2x_0 = ac - b^2x_1 + aby_1 \qquad \text{[gather } x_0 \text{ terms]}$$
$$-x_0(a^2 + b^2) = ac - b^2x_1 + aby_1$$
$$x_0 = \frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}$$

Substitute x_0 into line l to determine y_0.

$$y_0 = \left(\frac{-a}{b}\right)\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}\right) - \frac{c}{b}$$
$$= \frac{-abx_1 + a^2y_1 + \frac{a^2c}{b} - \frac{c(a^2+b^2)}{b}}{a^2 + b^2}$$
$$= \frac{-abx_1 + a^2y_1 + \frac{a^2c}{b} - \frac{a^2c}{b} - \frac{b^2c}{b}}{a^2 + b^2}$$
$$y_0 = \frac{-abx_1 + a^2y_1 - bc}{a^2 + b^2}$$

c. Now that we know the co-ordinates of $M(x_0, y_0)$, we can calculate d_{MP} based on the regular distance formula, $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$. Due to the algebraic complexity of x_0 & y_0, we will break this into the smaller steps prescribed by the question:

i.

$$x_1 - x_0 = \frac{a^2x_1 + b^2x_1}{a^2 + b^2} - \frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}$$
$$= \frac{a^2x_1 + b^2x_1 - b^2x_1 + aby_1 + ac}{a^2 + b^2}$$
$$= \frac{a^2x_1 + aby_1 + ac}{a^2 + b^2}$$
$$x_1 - x_0 = \frac{a(ax_1 + by_1 + c)}{a^2 + b^2}$$

Similarly, we can calculate $y_1 - y_0$.

ii.

$$y_1 - y_0 = \frac{a^2y_1+b^2y_1}{a^2+b^2} - \frac{-abx_1+a^2y_1-bc}{a^2+b^2}$$
$$= \frac{a^2y_1+b^2y_1+abx_1-a^2y_1+bc}{a^2+b^2}$$
$$= \frac{b^2y_1+abx_1+bc}{a^2+b^2}$$
$$y_1 - y_0 = \frac{b(ax_1+by_1+c)}{a^2+b^2}$$

d. Now we can (at last!) put all the pieces together:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$
$$= \sqrt{\frac{a^2(ax_1+by_1+c)^2}{(a^2+b^2)^2} + \frac{b^2(ax_1+by_1+c)^2}{(a^2+b^2)^2}}$$
$$= \sqrt{\frac{(a^2+b^2)(ax_1+by_1+c)^2}{(a^2+b^2)^2}}$$
$$= \sqrt{\frac{(ax_1+by_1+c)^2}{a^2+b^2}}$$
$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} \qquad \text{Q.E.D.}$$

e. By using this newly-derived formula to calculate the answer we determined for question 1, we can verify that our formula is correct and see whether it saves us time and effort when compared to the original method.

$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$
$$= \frac{|3(3)+1(4)-5|}{\sqrt{3^2+1^2}}$$
$$= \frac{9+4-5}{\sqrt{10}}$$
$$= \frac{8}{\sqrt{10}}$$
$$= \frac{8\sqrt{10}}{10}$$
$$d = \frac{4\sqrt{10}}{5} \text{ units}$$